Theorem (Markov inequality) Let $X$ be a non-negative random variable on a probability space with $0 < EX < \infty$. Then for all $t > 0$

$$P(X \geq tEX) \leq \frac{1}{t}$$

Proof

$$EX = \int X \, dP = \int X \, dP \geq tEX \int_{[X \geq tEX]} 1 \, dP = \int_{X \geq tEX} 1 \, dP = tEX \cdot P(X \geq tEX)$$

and so

$$P(X \geq tEX) \leq \frac{1}{t} \quad \square$$

Corollary $P(X \geq \frac{EX}{k}) \leq \frac{EX}{k}$ for every $k > 0$ and every non-negative random variable $X$.

Pf. Let $k = tEX$. $\square$
In weekly problem sets we have seen two different constructions of triangle-free graphs of arbitrary large chromatic number. It was mentioned that there exist graphs of arbitrary large chromatic number with no short cycles. We may prove it using random graphs. The proof is nonconstructive. There are constructive proofs, but for given \( k, l \) the graphs are much bigger than what random graphs give.

**Theorem (Erdős's 1959)** For every two integers \( k, l \) there exists a graph \( G \) with \( \chi(G) \geq k \) and no cycles of length at most \( l \).

**Proof.** We will consider \( G(n, p) \), where \( p = p(n) \) will be determined later. We first prove that for a suitable choice of \( p \), with high probability a random graph \( G \in G(n, p) \) has few short cycles. For the purpose of this calculation, let \( X_i(G) \) be the number of cycles in \( G \) of length exactly \( i \). Then

\[
X(G) := \sum_{i = 3}^{l} X_i(G)
\]

is the number of cycles in \( G \) of length at most \( l \).
$$EX = \sum_{i=3}^{l} E_{X_i} = \sum_{i=3}^{l} \sum_{|A|=i}^{i} \sum_{\sigma}^{A \subseteq \{1,2,\ldots,n\}} P[\sigma \text{ determines a cycle}] =$$

$$= \sum_{i=3}^{l} \binom{n}{i} \frac{1}{2^{(i-1)!}} \quad p^i = \sum_{i=3}^{l} \frac{n^i}{i \cdot (n-i)!} \quad \frac{1}{2} \cdot p^i = \frac{1}{2} \sum_{i=3}^{l} \frac{(np)^i}{i}$$

(The last summation is over all cyclic orderings of the set \(A\))

Now it seems sensible to choose \(p = p(n)\) so that \(np = n^\theta\) for some \(\theta > 0\)

$$= \frac{1}{2} \sum_{i=3}^{l} \frac{n^{\theta \cdot i}}{i} \leq \frac{1}{2} \cdot n^{\theta l} \quad \text{for } n \text{ sufficiently large}.$$  

If we choose \(\theta < \frac{1}{l}\), then \(EX = o(n)\). By Markov's inequality,

$$P[X \geq \frac{n}{2}] \leq \frac{2EX}{n} = o(1)$$

and so for all sufficiently large \(n\) \(P[X \geq \frac{n}{2}] < \frac{1}{2}\). That is,

(1) the probability that \(G \in G(n,p)\) has \(\geq \frac{n}{2}\) cycles of length \(\leq l\) is strictly less than \(\frac{1}{2}\).
To get a lower bound on $\chi(G)$ we will use the inequality $\chi(G) \leq \alpha(G) \leq n$. Thus we need an upper bound on $\alpha(G)$; in other words we need upper bound on $P[\chi(G) \geq t]$. 

$$P[\chi(G) \geq t] = P[G \text{ has an independent set of size } t] \leq$$

$$\leq \sum_{A \subseteq \{1,2,\ldots, n\}} P[A \text{ is independent in } G] = \binom{n}{t} (1 - p)^{\binom{t}{2}} \leq n^t (1 - p)^t \leq n^t e^{-p \binom{t}{2}} = \left[ n e^{-p \left(\frac{t}{2}\right)} \right]^t = o(1) \quad \text{if } t = \frac{3}{p} \log n$$

$$\uparrow \text{ using } 1 + x \leq e^x$$

Thus for $t = \frac{3}{p} \log n$ and $n$ sufficiently big we have

$$P[\chi(G) \geq t] \leq \frac{1}{2}$$

Thus by (1) and (2), there exists a graph on $n$ vertices with $\leq \frac{1}{2}$ cycles of length $\leq l$ and $\alpha(G) \leq t$. By deleting one vertex from each cycle of length $\leq l$ we deduce that $G$ has an induced subgraph $G'$ on at least $\frac{1}{4}$ vertices with no cycle of length $\leq l$ and $\alpha(G') \leq t$. Now
\[
\gamma(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{w/2}{t} \geq \frac{w/2}{t} = \frac{n}{3p \log n} = \frac{n}{6n^{1-\theta} \log n}
\]

\[
= \frac{n}{6 \log n} \rightarrow \infty
\]

and so for \( n \) sufficiently large \( \gamma(G') \geq k \), as desired. \( \Box \)
Our next objective is to count $K_4$ subgraphs. It is reasonable to expect that if $p$ is very small (as a function of $n$), then a.e. graph in $G(n,p)$ will have no $K_4$ subgraph, while for large $p$ a.e. graph in $G(n,p)$ will have a $K_4$ subgraph. What is interesting is that this change ("phase transition") appears suddenly and the probability of the appearance of a $K_4$ subgraph goes from 0 to 1 rather rapidly.

For $A = \{1, 2, \ldots, n\}$ with $|A| = 4$, let

$$X_A(G) = \begin{cases} 1 & \text{if } A \text{ is a clique of size 4 in } G \\ 0 & \text{otherwise} \end{cases}$$

Then the number of $K_4$ subgraphs of $G$ is

$$X(G) := \sum_{|A| = 4} X_A(G)$$

$$A \subseteq \{1, 2, \ldots, n\}$$

We have

$$EX = \sum_{|A| = 4} EX_A = \binom{n}{4} p^6, \quad \text{and by Markov}$$

$$P[G \text{ has a } K_4 \text{ subgraph}] = P[X \geq 1] \leq EX \leq n^4 p^6$$
For $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$ we define $f \ll g$ to mean that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

Thus we have

If $p(n) \ll \frac{1}{n^{1/3}}$, then a.e. graph in $G(n, p)$ has no $K_4$ subgraph.

Conversely, we will prove that if $p(n) \gg \frac{1}{n^{1/3}}$, then a.e. graph in $G(n, p)$ has a $K_4$ subgraph. That will need a different method, because it is not implied by $EX \rightarrow \infty$. We will need to bound the probability that $X$ deviates from $EX$. 