Lemma 2.13 Let $G$ have $\geq 4$ vertices and no $K_5$ or $K_{3,3}$ subdivision. Assume that adding an edge joining a pair of non-adjacent vertices creates a $K_5$ or $K_{3,3}$ subdivision. Then $G$ is 3-connected.

Proof. Induction on $|V(G)|$.

Exercise: Prove $G$ is 2-connected.

To show $G$ is 3-connected suppose that $G \setminus \{u,v\}$ is disconnected. Thus $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{u,v\}$.

\begin{center}
\includegraphics[width=0.5\textwidth]{example_diagram}
\end{center}
Claim \( u \cup v \) or \( K_5 \) or \( K_{3,3} \) Subdivision exists by 2-connecting.

So let \( u \in V(G_1) \cap V(G_2) \). By induction both \( G_1 \) and \( G_2 \) are either 3-connected or a triangle. By Lemma 2.10, \( G_i \) is a plane graph. Pick \( w_i \in V(G_i) - \{u, v\} \) such that \( u, v, w_i \) belong to the same face.
By hypothesis, $G + w, w_2$ has a $K_5$ or $K_{3,3}$ subdivision, say $K$.

All except possibly one of the branch-vertices of $K$ belong to the same $G_i$, say $G_1$. It follows that some branch-vertex of $K$ belongs to $G_2$, for otherwise there would be a $K_{3,3}$ subdivision in $G_1$.

Let $G'_1 = G_1 + \nu$, new vertex adjacent to $w, w_2, w_3$.

Then $G'_1$ is a plane graph. $K$ can be converted to a $K_5$ or $K_{3,3}$ subdivision in $G'_1$, a contradiction.
\( cr(G) \) := minimum number of crossings in a drawing of \( G \) in the plane (in which crossings are allowed)

More precisely, our "drawings" now allow edges to intersect but:

1. \( |e \cap e'| \) is finite for distinct \( e, e' \in E(G) \)
2. each point of \( \mathbb{R}^2 \) belongs to \( \leq 2 \) edges

So we have

\[
\sum \text{ against the number of crossings is } \sum |e \cap e'|
\]

Theorem (Ajtai, Chvátal, Newborn, Szemerédi)

Let \( G \) be a simple graph. Then

\[
cr(G) = \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2} - |V(G)|
\]

Lemma \( cr(G) \geq |E(G)| - 3|V(G)| \)

Proof. Remove edges to get planar graph on \( n \) vertices \& \( \geq 3n \) edges
Proof of lemma (Szekely). Let $c$ be the crossing number of $G$, let $n = |V(G)|$, $m = |E(G)|$.

We have $m \geq 4n$, for otherwise the RHS is negative.

Let $p \in (0, 1)$, TBD. Choose a random subset $V \subseteq V(G)$ by picking each vertex independently at random with probability $p$. The expected number of

- vertices is $pn$
- edges is $p^2m$
- crossings is $p^4c$

By Lemma

$$p^4c \geq p^2m - 3pn \implies c \geq \frac{m}{p^2} - 3 \frac{n}{p^3}$$

Choose $p = \frac{\sqrt[4]{n}}{m} (< 1)$ because $m \geq 4n$ we get

$$c \geq \frac{m^3}{64n^2}$$

$\square$
Let $I(n,m)$ be the maximum number of possible incidences between $n$ points and $m$ lines in the plane. That is, the maximum over all sets $P \subseteq \mathbb{R}^2$ and sets of lines $L$ such that $|P| = n$, $|L| = m$, of

$$\left| \left\{ (p, l) : p \in P, l \in L, p \in l \right\} \right|.$$ 

**Example**  

$I(3,3) \geq 6$
Theorem (Szemerédi-Trotter) For all \( n, m \geq 1 \)

\[ I(n, m) = O(n^{2/3} m^{2/3} + n + m) \] and the bound is asymptotically tight.

Proof (Székely) Let \( P, L \) be a system of points and lines realizing \( I(n, m) \). Define a topological graph (= graph drawn with crossings) \( G \) by \( V(G) = P \) and \( E(G) \) subsets of lines in \( L \) connecting consecutive points.

A line \( L \in L \) containing \( k \) points contributes \( k\)-edges.
\[ I(n, m) = |E(G)| + m \]

\[ I(n, m) = \sum_{L \in \mathcal{Y}} \#L = \sum_{L \in \mathcal{Y}} \left( \text{# edges of } G \text{ contributed by } L \right) \]

\[ \text{# points of } P \text{ on } L \]

\[ = |E(G)| + m. \]

We have

\[ \binom{m}{2} \geq cr(G) \geq \frac{1}{64} \frac{|E(G)|^3}{n^2} - n \]

\[ |E(G)|^3 \leq 32 m^2 n^2 + 64 n^3 \]

\[ I(n, m) = |E(G)| + m \leq \Theta \left( m^{2/3} n^{2/3} + n + m \right) \]