Proof: If $G$ is disconnected, then $\tau(G) = 0 = \det L(k)$.

If $G = \circlearrowleft$, then $\tau(G) = 1 = \det L(k)$.

By induction on $|E(G)|$. Recall

$$\tau(G) = \tau(G \setminus e) + \tau(G/e)$$

Enough to show

$$\det L_G(w) = \det L_{G \setminus e}(w) + \det L_{G/e}(w)$$

where $e = uv$ and $w$ is the new vertex of $G/e$.

$$L_G(w) = L_{G \setminus e}(w) + E_{ww}$$

$$\det L_G(w) = \det [L_{G \setminus e}(w) + E_{ww}] = \det L_{G \setminus e}(w) +$$

$$+ \det L_{G \setminus e}(w,v) = \det L_{G \setminus e}(w) + \det L_{G/e}(w)$$

This proves (*) and hence the theorem. \qed
\[
\text{What is } \det (M + E_{11})? \quad \text{What } v = 1.
\]
\[
\det (M + E_{11}) = \sum_{\sigma} \text{sgn} (\sigma) \prod_{i=1}^{n} (M + E_{\sigma(i)}) =
\]
\[
= \sum_{\sigma; \sigma(1) = 1} \text{sgn} (\sigma) (M_{11} + 1) \prod_{i=2}^{n} M_{i \sigma(i)} + \sum_{\sigma; \sigma(1) \neq 1} \prod_{i=1}^{n} M_{i \sigma(i)} =
\]
\[
= \sum_{\sigma} \text{sgn} (\sigma) \prod_{i=1}^{n} M_{i \sigma(i)} + \sum_{\sigma; \sigma(1) = 1} \text{sgn} (\sigma) \prod_{i=2}^{n} M_{i \sigma(i)} =
\]
\[
= \det (M) + \det M(1)
\]
Theorem (Cayley) \( \tau(K_n) = n^{n-2} \). In other words, there are exactly \( n^{n-2} \) trees with vertex set \{1, 2, \ldots, n\}.

Proof. By Kirchhoff's Matrix Tree Theorem

\[
\tau(K_n) = \det \begin{bmatrix}
 n-1 & -1 & -1 \\
 -1 & n-1 & -1 \\
 -1 & -1 & n-1 \\
\end{bmatrix}^{n-1} = \det \begin{bmatrix}
 1 & 1 & \cdots & 1 \\
 -1 & n-1 & \cdots & -1 \\
 -1 & -1 & \cdots & n-1 \\
\end{bmatrix} = \det \begin{bmatrix}
 1 & 1 & \cdots & 1 \\
 0 & 0 & \cdots & 0 \\
\end{bmatrix} = 0
\]

Add row 2, \ldots, \( n-1 \) to row 1

\[
\tau(K_n) = n^{n-2}
\]

\( \square \)
Def: Let $G$, $H$ be multigraphs. A mapping $f: V(G) \rightarrow V(H)$ is an isomorphism if it is a bijection and for any two vertices $u, v \in V(G)$, the number of edges with ends $uv, vw$ in $G$ is the same as the number of edges with ends $f(u), f(v)$ in $H$.

Necessary conditions: $|V(H)| = |V(G)|$, $|E(H)| = |E(G)|$, same degree sequence (needs definition).

Examples:

adjacent deg 3 vertices

vertices deg 3 not adjacent

bipartite

not bipartite
The max-flow min-cut theorem

A directed graph or digraph is a pair $D = (V, E)$, where $V$ is a finite set and $E \subseteq V \times V$.

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The max-flow min-cut theorem

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The max-flow min-cut theorem

A directed graph or digraph is a pair $D = (V, E)$, where $V$ is a finite set and $E \subseteq V \times V$.
A network is a quadruple $N = (D, s, t, c)$, where $D$ is a digraph, $s, t \in V(D)$ are distinct, and $c : E(D) \to [0, \infty]$.

$\delta^+(X) = \{ e \in E(D) : e$ has tail in $X,$ head in $V(D) - X \}$

$\delta^-(X) = \{ e \in E(D) : e$ has head in $X,$ tail in $V(D) - X \}$

$\delta^+(\{ s \}) = : \delta^+(s),$ $\delta^-(\{ t \}) = : \delta^-(t)$

If $f : E(D) \to \mathbb{R}$, then

$f^+(X) = \sum_{e \in \delta^+(X)} f(e),$ $f^-(X) = \sum_{e \in \delta^-(X)} f(e)$

A flow in $N$ is a mapping $f : E(D) \to \mathbb{R}$ such that

(i) $0 \leq f(e) \leq c(e)$ $\forall e \in E(D)$ capacity constraint

(ii) $f^+(v) = f^-(v)$ $\forall v \in V(D) - \{ s, t \}$ conservation condition
Lemma. If $f$ is a flow in a network $N = (D, \delta, c)$ and $X \subseteq V(D)$ with $s \in X$, $t \notin X$, then

$$f^+(A) - f^-(A) = f^+(X) - f^-(X)$$

net outflow out of $A$

Proof. $f^+(v) = f^-(v)$ $\forall v \in V(D) - \{s, t\}$

Sum over all $v \in X - \{s\}$

$$\sum_{e \in \delta^+(X-\{s\})} f(e) = \sum_{e \in \delta^-(X-\{s\})} f(e)$$

$$f^+(X) + f^-(A) + \sum_{e \in \delta^+(X-\{s\})} f(e) = f^-(X) + f^+(A) + \sum_{e \in \delta^-(X-\{s\})} f(e)$$

e has both ends in $X-\{s\}$

e has both ends in $X-\{s\}$