Open problem: Is there a function $f: \mathbb{N} \to \mathbb{N}$ such that for every graph $G$, for all $a, b \in V(G)$ if $G$ is $f(k)$-connected, then there exists an $a$-$b$ path $P$ such that $G \setminus V(P)$ is $k$-connected?

Known that $f(2)$ exists.
$G$ is $k$-connected if it has $\geq k+1$ vertices and $G \setminus X$ is connected $\forall X \subseteq V(G)$ $|X| < k$.

$G$ is $k$-edge-connected if $G \setminus F$ is connected $\forall F \subseteq E(G)$ with $|F| < k$.

$kappa$ refers to the connectivity of $G$, denoted by $\kappa(G)$, is the maximum integer $k$ s.t. $G$ is $k$-connected.

The edge-connectivity of $G$, denoted by $\kappa'(G)$, is the maximum integer s.t. $G$ is $k$-edge-connected.

$\exists$ graphs with $\kappa(G)=1$, $\kappa'(G)$ big.
Proposition \( x(G) \leq x'(G) \leq \delta(G) \) \\
\text{Proof} \quad x'(G) \leq \delta(G) \text{ clear} \\
\quad x(G) \leq x'(G) \\
Let \( k := x'(G) \), and let \( F \subseteq E(G) \) be such that \\
\( |F| = k \) and \( G \setminus F \) is disconnected.

\[ x_0 \quad x_1 \quad \cdots \quad x_k \]
\[ y_0 \quad y_1 \quad \cdots \quad y_k \]

Need to find \( X \subseteq V(G) \) \(|X| \leq k \) s.t. \( G \setminus X \) is disconnected. \emph{WLOG} \( G \setminus \{x_0, y_0, y_1, \ldots, y_k\} \) is connected.

Similarly, \emph{WLOG} \( G \setminus \{y_0, y_1, \ldots, y_k\} \) is connected.

So \( V(G) = \{x_0, x_1, \ldots, x_k, y_0, y_1, \ldots, y_k\} \).
 Sikh $x_1 = x_2 = \ldots = x_p \neq x_j \quad \forall j = p + 1, \ldots, k$

\[ \left\{ \begin{array}{c}
\{ x_1 = x_2 = \ldots = x_p \\
0 \times_{p+1} \\
0 \\
x_k 
\end{array} \right. \]

\[ \begin{array}{c}
o_{q_1} \\
o_{q_2} \\
\vdots \\
0 \\
0 
\end{array} \]

- WNA $G$ is simple
  \[ \deg (x_i) \leq \# \text{hbrs among } x_i \text{s } + \# \text{hbrs among } y_i \text{s } \leq \]
  \[ \leq k - p + p = k \]

Let $N$ be the set of hbrs of $x_1$. Then $|N| \leq k$.

If $G \setminus N$ is disconnected, then done.

Otherwise $V(G) = N \cup \{ x_1 \} \implies |V(G)| \leq k + 1$

\[ \implies \chi(G) \leq k \]

\[ \square \]
Block structure of (connected) graphs

\[ G[E_1], G[E_2] \text{ have only} \]

\[ \text{or in common} \]

Ref. A \underline{cutvertex} in a multigraph \( G \) is a vertex \( v \)

such that \( E(G) \) can be partitioned into disjoint

non-empty sets \( E_1, E_2 \) such that both \( E_1 \) and \( E_2 \) include

an edge incident with \( v \).

Ex1:

\[ E_1 \]

\[ E_2 \]

Ex2 \( G \setminus v \) disconnected
Def. A block is a graph with no cut vertex.

Examples: \( \circ, \bullet, \circ, \bullet, \circ \)

loopless & 2-connected

Def. A block of a multigraph \( G \) is a maximal submultigraph that is a block.

Proposition (i) Every multigraph is a union of its blocks.
(ii) If \( B_1, B_2 \) are distinct blocks of \( G \), then
\[ |V(B_1) \cap V(B_2)| \leq 1 \]
and if \( v \in V(B_1) \cap V(B_2) \)
then \( v \) is a cut vertex.

Proof. (i) immediate
(ii) Suppose \( u, v \in V(B_1) \cap V(B_2) \), \( u \neq v \)
\[ \Rightarrow B_1 \cup B_2 \text{ is a block} \]
If $B_1, B_2$ are 2-connected $\Rightarrow B_1 \cup B_2$ 2-connected

Proof: Let $z \in V(B_1 \cup B_2)$. Then $B_1 \setminus z, B_2 \setminus z$ are connected, and they intersect $\Rightarrow (B_1 \setminus z) \cup (B_2 \setminus z)$ is connected

$(B_1 \cup B_2) \setminus z$

If $z \in V(B_1) \cap V(B_2)$
Given a multigraph, define a graph \( F \) as follows:

\[ V(F) = B \cup C \]

all blocks of \( G \)

all cut-vertices of \( G \)

\( B \) is adjacent to \( C \) if \( c \in V(B) \)

**Theorem** \( F \) is a forest, and if \( G \) is connected, then it is a tree.

**Def.** This is called the block structure of a graph.

**Proof.** Exercise
Theorem (Ear structure of 2-connected graphs)

Let $G$ be a 2-connected graph. Then $G$ can be written as $G = G_0 \cup G_1 \cup \ldots \cup G_k$, where

(i') $G_0$ is a cycle, and
(ii') for $i = 1, 2, \ldots, k$, $G_i$ is a path with both ends in $G_0 \cup \ldots \cup G_{i-1}$ and otherwise disjoint from it.

Proof. If cycle. We can pick $G_0, G_1, \ldots, G_k$ satisfying (i') and (ii') with $k$ maximum.