\( G \text{ \( k \)-connected} \implies \delta(G) = \min \text{ degree} \geq k \)

\[ K_{10^4+1} \quad \text{and} \quad K_{10^6+1} \]

**Thm (Mader)** Every graph of min degree at least \( 4k \) has a \( k \)-connected subgraph.
Thoughts about a possible proof

1. Replace min degree by average degree

2. Let's try to prove \( n := |V(G)| \)

\[ |E(G)| \geq \alpha n + \beta + 1 \implies G \text{ has } k\text{-connected subgraph} \]

\[
\begin{align*}
\bullet & \quad \bullet \\
G & \quad \quad \quad \quad 3 \quad \quad \\
\bullet & \quad \bullet \\
G_1 & \quad \quad \quad \quad \quad G_2
\end{align*}
\]

Is it possible that for some values of \( \alpha, \beta \):

(1) If \( G \) is counterexample, then so is \( G_1 \) or \( G_2 \)?

Know: \( |E(G)| \geq \alpha n + \beta + 1 \) \( \implies G \) has no \( k \)-connected subgraph

\( G_i \) is not a counterexample & \( G_i \) has no \( k \)-connected subgraph (because \( G \) has none) \( \implies |E(G_i)| \leq \alpha n + \beta \)

\( n := |V(G_i)| \) Need \( |V(G_i)| \geq ck \) (i = 1, 2)

Fails from \( \Sigma(G_i) \geq ck \), which we proved last.
\[ |E(G)| \leq |E(G_1)| + |E(G_2)| \leq \alpha n_1 + \beta + \alpha n_2 + \beta \leq \alpha (n+k) + 2\beta = \alpha n + \beta + 2k + \beta \leq \alpha n + \beta \ ? \leq 0 ? \]

If \( \alpha k + \beta \leq 0 \), then above computation gives a contradiction, and hence proves \((\star)\)

Let's pick \( \beta := -2k \). Let \( \alpha = ck \). Thus we are trying to prove

\[(\star \star) \quad |V(G)| = ck \quad \Rightarrow \quad \text{G has a} \quad k\text{-connected subgraph}

For suitable \( c \), there is no graph \( G \) with \( |V(G)| = ck \)

and \( |E(G)| \geq ckn - ck^2 + 1 \). If \( |V(G)| = ck \), then

\[(ck)^2 - ck^2 + 1 \leq |E(G)| \leq \binom{ck}{2} = \frac{1}{2}(ck)^2 \]

\[\frac{1}{2}(ck)^2 < c k^2 \]

If \( c = 2 \), then the graph \( c \leq 2 \) does not exist.
If $G$ is a minimum counterexample to $(\star \star)$

then the

$\Delta(G) \geq ck$.

If $|V(G)| > ck$

If $\nu$ has degree $\leq ck$, then

$$|E(G\setminus \varepsilon)| \geq ckn - ck^2 + 1 - ck =$$

$$= ck(n-1) - ck^2 + 1$$

$$\frac{|V(G\setminus \varepsilon)|}{|V(G\setminus \varepsilon)|}$$

$\Rightarrow G \setminus \nu$ is a smaller counterexample.
Matching

A matching in $G$ is a set $M \subseteq E(G)$ such that every vertex of $G$ is incident with at most one edge of $M$.

$M$ saturates $v \in V(G)$ or $v$ is saturated by $M$ if $v$ is incident with an edge of $M$.

A matching is perfect if it saturates every vertex.

A maximum matching is a matching $M$ in $G$ such that there is no matching $M'$ with $|M'| > |M|$.

A maximal matching is a matching $M$ such that there is no matching $M'$ with $M \subseteq M'$. 
An \textit{M-alternating path} is a path s.t. edges in \( M \) and edges not in \( M \) alternate along \( P \).

An \textit{M-augmenting path} is an \( M \)-alternating path that starts and ends in an \( M \)-unsaturated vertex.

Let \( M' := M + E(P) \). Then \( M' \) is a perfect matching with \( |M'| > |M| \).
Thm (Berge) A matching \( M \) in \( G \) is maximum if and only if there is no \( M \)-augmenting path.

Proof. \( \Rightarrow \) done

\( \Leftarrow \) Assume \( M \) is not maximum.

Let \( M' \) be a matching with \( |M'| > |M| \).

Look at the graph \( H \) with \( V(H) = V(G) \),

\( E(H) = M \triangle M' \). Then \( \Delta(H) \leq 2 \)

The components of \( H \) are:

- even cycles (same # of edge of \( M \) & \( M' \))
- paths

\( \Rightarrow \) I component \( P \) of \( H \) that has more edges in \( M' \) than in \( M \). That's an \( M \)-augmenting path. \( \square \)