A bipartite graph, bipartition \((A, B)\)

A matching \(M\) is a **complete matching from** \(A\) to \(B\) if it saturates every vertex of \(A\).

\[
\begin{array}{c|c|c}
A & \mathcal{X} & B \\
\hline
\end{array}
\]

If \(|A| = |B|\) then \([\text{complete from } A \text{ to } B \Rightarrow \text{perfect}]\)

**Obstruction:**

\[\begin{array}{c|c|c}
A & S & A \\
\hline
\end{array}\]

\[\begin{array}{c|c|c}
\mathcal{N}(S) & |\mathcal{N}(S)| < |S| \\
\hline
\end{array}\]

\(\Rightarrow \) complete matching \(A\) to \(B\)
Thm (Hall) A bipartite graph with bipartition \((A,B)\) has a complete matching from \(A\) to \(B\) if and only if \(|N(S)| \geq |S|\) for every \(S \subseteq A\).

\[ N(S) = \{ v \in S : \exists x \text{ adjacent to } v \text{ in } S \} \]

Proof #1 Using Menger's thm

\(\Rightarrow\) already done

\(\Leftarrow\): If there exist \(|A|\) disjoint paths from \(A\) to \(B\), then their edge-sets form a complete matching from \(A\) to \(B\).

Thus \(\forall A \exists B \text{ disjoint } A-B \text{ paths. By Menger's theorem } \exists X \subseteq V(G) \text{ such that } G \setminus X \text{ has no } A-B \text{ path and } |X| < |A| \). Let \(S := A - X\).

\[ |N(S)| \leq |X \cap B| = |X| - |X \cap A| < |A| - |X \cap A| \]

\(-|X \cap A| = |S|\), a contradiction. \(\square\)
**Proof #2** From first principles

\[\Rightarrow \text{ only}\]

Case 1: \(|N(S)| > |S|\) for every \(\emptyset \neq S \subseteq A\).

- \(A\)
- \(B\)

Pick \(v \in A\), pick a nbr \(w\) of \(v\), apply induction to \(G \setminus \{u, v\}\).

Case 2: \(|N(S)| = |S|\) for some \(\emptyset \neq S \subseteq A\).

- \(S\)
- \(A\)
- \(B\)

Let \(G_1 := G \setminus \text{SUN}(S)\)

\(G_2 := G \setminus \text{SUN}(S)\)

Apply induction to \(G_1\) and \(G_2\).

\(G_1\) clearly satisfies the induction hypothesis. To see that \(G_2\) satisfies the induction hypothesis for \(L \subseteq A - S\), look at \(N_G(L \cup S)\).

\(|N_G(L \cup S)| \geq |L \cup S| = |L| + |S|\) \(\Rightarrow |N_{G_2}(L)| \geq |L|

\(\Rightarrow |N_G(S)| + |N_{G_2}(L)|\)
When does a (not necessarily bipartite) graph have a perfect matching?

Let \( o(H) \) = \# of odd components of \( H \)

If \( o(G \setminus X) > |X| \), then \( G \) has no perfect matching.

Thm (Tutte 1956) \text{TNCAS}

A graph \( G \) has a perfect matching if and only if

\[ o(G \setminus X) \leq |X| \text{ for every } X \subseteq V(G). \]
Def

Let $M$ be a matching in $G$. A cycle $C$ in $G$ of length $2k+1$ containing $k$ edges of $M$ is called an $M$-blossom. Let $G/C$ denote the graph obtained from $G$ by contracting all edges of $C$ and deleting all loops and parallel edges.

Lemma Let $M$ be a matching in $G$, and let $C$ be an $M$-blossom in $G$. Let $G' = G/C$ and $M' = M - E(C)$. If $M$ is a maximum matching in $G$ then $M'$ is a maximum matching in $G'$. 
Proof. Suppose not. \( E \) \( M \)-augmenting path \( P' \) in \( G' \). Will exhibit an \( H \)-augmenting path in \( G \). Let \( v \) be the new vertex of \( G' \). \( VHA \) \( w \in V(P') \) for \( v \). \( P' \) is as desired.

The vertex \( w \) divides \( P \) into \( P_1 \) and \( P_2 \). Let \( u \) be the ends of \( P \). \( VHA \) by symmetry that the edge of \( P_2 \)
incident edge is in $M$. Then in $G$ the path $P_2$ becomes a path from $v$ to the tip of the blossom, and $P_1$ becomes a path from $u$ to the blossom. Follow $P_1$ from $u$ to $u' \in V(C)$, then follow $C$ along the even path from $u'$ to the tip, and then follow $P_2$. That gives an $M$-augmenting path in $G$. $\square$.