An edge-coloring of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The edge-chromatic number or chromatic index of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

Observation. $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq \ldots$
An **edge-coloring** of $G$ is a mapping $c : E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

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**Theorem (Vizing)** For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

| $E(G)$ | $= 3k$ |
|--------|
| $\Delta(G)$ | $= 2k$ |
| $\chi'(G)$ | $= 3k$ |
An edge-coloring of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The edge-chromatic number or chromatic index of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

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**Theorem (Vizing)** For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

![Graph](image.png)

$|E(G)| = 3k$

$\Delta(G) = 2k$

$\chi'(G) = 3k$

**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 
\[ s \text{ missing} \]

\[ x \quad \rightarrow \quad y_1 \quad \text{missing} \]
An **edge-coloring** of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

**Observation.**  $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$

**Theorem** (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

\begin{align*}
|E(G)| &= 3k \\
\Delta(G) &= 2k \\
\chi'(G) &= 3k
\end{align*}

**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 
An edge-coloring of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The edge-chromatic number or chromatic index of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

**Observation.** $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$

**Theorem** (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

![Diagram](image)

$|E(G)| = 3k$

$\Delta(G) = 2k$

$\chi'(G) = 3k$

**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 
An **edge-coloring** of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

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**Theorem** (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

![Graph](image)

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Construct this for as long as $t_1, \ldots, t_k$ are pairwise distinct and the edges $xy_i$ exist.

**Case 1.** $t_k$ is missing at $x$.

**Case 2.** $t_k = t_j$ for some $j = 1, 2, \ldots, k - 1$. 
An **edge-coloring** of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

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**Example.**

![Diagram](image)

\[|E(G)| = 3k\]
\[\Delta(G) = 2k\]
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**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 

15
Construct this for as long as $t_1, \ldots, t_k$ are pairwise distinct and the edges $xy_i$ exist.

**Case 1.** $t_k$ is missing at $x$. Color $xy_i$ using $t_i$ for $i = 1, 2, \ldots, k$.

**Case 2.** $t_k = t_j$ for some $j = 1, 2, \ldots, k - 1$. 
An edge-coloring of $G$ is a mapping $c: E(G) \to S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

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**Theorem** (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

![Graph Example](image)

$|E(G)| = 3k$

$\Delta(G) = 2k$

$\chi'(G) = 3k$

**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 
Construct this for as long as \( t_1, \ldots, t_k \) are pairwise distinct and the edges \( xy_i \) exist.

**Case 1.** \( t_k \) is missing at \( x \). Color \( xy_i \) using \( t_i \) for \( i = 1, 2, \ldots, k \).

**Case 2.** \( t_k = t_j \) for some \( j = 1, 2, \ldots, k - 1 \).

Let \( H \) be the subgraph of \( G \) consisting of edges colored \( s \) or \( t_k \).

Notice that if we swap \( s \) and \( t_k \) in any component of \( H \) we get a valid edge-coloring.

Swap \( s, t_k \) in the component of \( H \) containing \( y_k \). Then \( s \) will be missing at \( y_k \). Color \( xy_k \) using \( s \), color \( xy_i \) using \( t_i \) for \( i = 1, 2, \ldots, k - 1 \).
An **edge-coloring** of $G$ is a mapping $c: E(G) \rightarrow S$, where $S$ is some set, such that $c(e) \neq c(f)$ for every two adjacent edges $e, f$. It is a $k$-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of $G$, denoted by $\chi'(G)$, is the least integer $k$ such that $G$ has a $k$-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi(L(G))$.

**Observation.** $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$

**Theorem** (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

**Example.**

```
                    k
                     |
                     |
  a ----------------- b
                      |
                      |
  y_1 ----------------- y
```

$|E(G)| = 3k$

$\Delta(G) = 2k$

$\chi'(G) = 3k$

**Proof.** By induction on $|E(G)|$. By induction we can color all but one edge of $G$ using $\Delta(G) + 1$ colors. Let $xy_1$ be the uncolored edge. For every $v \in V(G)$ there is a “missing color” at $v$; that is, a color not used by any edge incident with $v$. 
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Let $H$ be the subgraph of $G$ consisting of edges colored $s$ or $t_k$.

Notice that if we swap $s$ and $t_k$ in any component of $H$ we get a valid edge-coloring. Either ($x$ and $y_j$) or ($x$ and $y_k$) are not in the same component of $H$.

**Case 2a.** $x, y_j$ are not in the same component of $H$

Swap $s$, $t_k = t_j$ in the component of $H$ containing $y_j$. Color $xy_j$ using $s$, color $xy_i$ using $t_i$ for $i = 1,2,\ldots,j - 1$.

**Case 2b.** Analogous (replace $j$ by $k$). \(\square\)
A communication model

A **discrete memoryless channel** has input alphabet $\Sigma$, output alphabet $\Sigma_{out}$

On input $a$ we might receive $a_1$ or $a_2$ or $a_3$ or \ldots
On input $b$ we might receive $b_1$ or $b_2$ or $b_3$ or \ldots
On input $c$ we might receive $c_1$ or $c_2$ or $c_3$ or \ldots

$a, b$ are confoundable if $a_j = b_i$ for some $i, j$.

**Example.** $\Sigma = \{a, b, c, d, e\}$

The graph indicates confoundable pairs.
Construct this for as long as $t_1, \ldots, t_k$ are pairwise distinct and the edges $xy_i$ exist.

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**Case 2b.** Analogous (replace $j$ by $k$). \hfill \Box
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A **discrete memoryless channel** has input alphabet $\Sigma$, output alphabet $\Sigma_{\text{out}}$

On input $a$ we might receive $a_1$ or $a_2$ or $a_3$ or $\cdots$
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The graph indicates confoundable pairs.
Two sequences \((x_1, ..., x_t), (y_1, ..., y_t)\) of elements of \(\Sigma\) are **confoundable** if \(\forall i = 1, 2, ..., t\) either \(x_i = y_i\) or \(x_i, y_i\) are confoundable.

**Objective.** A set of pairwise unconfoundable sequences of length \(t\)
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**Example 1.** Any sequence of \(a\)’s and \(c\)’s of length \(t\). That is a family of \(2^t\) pairwise unconfoundable sequences of length \(t\).
A communication model

A **discrete memoryless channel** has input alphabet \( \Sigma \), output alphabet \( \Sigma_{\text{out}} \)

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**Example 2.** A bigger family. Notice that

\[
\begin{align*}
&aa, bd, cb, de, ec \\
\end{align*}
\]

are pairwise unconfoundable. Take any sequence of these of length \(t/2\). That will give a collection of pairwise unconfoundable words of size \(5^{t/2} = (2^{\log_5^t})^{1/2} = 2^{(1/2 \log_5 5)t}\)
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On input $a$ we might receive $a_1$ or $a_2$ or $a_3$ or $\ldots$
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**Fekete’s lemma.** If \((a_t)_{t \geq 1}\) is a sequence of positive real numbers satisfying

\[
a_{s+t} \geq a_s + a_t
\]

then \(\lim_{t \to \infty} \frac{1}{t} a_t\) exists and is equal to \(\sup_{t \geq 1} \frac{1}{t} a_t\).
**Definition.** For graphs $K, L$ we define their product $K \boxtimes L$ by

$$V(K \boxtimes L) = V(K) \times V(L)$$

and

$$(k_1, \ell_1) \sim (k_2, \ell_2)$$ if

- $(k_1, \ell_1) \neq (k_2, \ell_2)$
- $k_1 = k_2$ or $k_1 \sim k_2$ in $K$
- $\ell_1 = \ell_2$ or $\ell_1 \sim \ell_2$ in $L$

**Example.** $K_2 \boxtimes K_2$

**Example.** $C_5 \boxtimes C_5$
Two sequences \((x_1, ..., x_t), (y_1, ..., y_t)\) of elements of \(\Sigma\) are **confoundable** if \(\forall i = 1, 2, ..., t\) either \(x_i = y_i\) or \(x_i, y_i\) are confoundable.

**Objective.** A set of pairwise unconfoundable sequences of length \(t\)

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\[ aa, bd, cb, de, ec \]

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- $k_1 = k_2$ or $k_1 \sim k_2$ in $K$ and
- $\ell_1 = \ell_2$ or $\ell_1 \sim \ell_2$ in $L$

**Example.** $K_2 \boxtimes K_2$

**Example.** $C_5 \boxtimes C_5$
Let $\gamma(G) := \chi(G^c) = \min \# \text{ of cliques covering the vertices of } G$.

**Observations:**

1. $\alpha(G_1 \boxtimes G_2) \geq \alpha(G_1)\alpha(G_2)$
2. $\alpha(C_5 \boxtimes C_5) = 5$
3. $\gamma(G_1)\gamma(G_2) \geq \gamma(G_1 \boxtimes G_2)$

**Proof of (3).** Let $K_1, ..., K_r$ be a cover by cliques of $G_1$.

Let $L_1, ..., L_s$ be a cover by cliques of $G_2$.

Then $\{K_i \times L_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ is a cover of $G_1 \boxtimes G_2$ by $rs$ cliques, as desired.
**Definition.** For graphs $K, L$ we define their product $K \boxtimes L$ by

$$V(K \boxtimes L) = V(K) \times V(L)$$

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$$(k_1, \ell_1) \sim (k_2, \ell_2) \text{ if }$$

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**Proof of (3).** Let \(K_1, \ldots, K_r\) be a cover by cliques of \(G_1\).
Let \(L_1, \ldots, L_s\) be a cover by cliques of \(G_2\).
Then \(\{K_i \times L_j: 1 \leq i \leq r, 1 \leq j \leq s\}\) is a cover of \(G_1 \boxtimes G_2\) by \(rs\) cliques, as desired.

Let \(G^t := G \boxtimes G \boxtimes \cdots \boxtimes G\) \((t\) times). The **Shannon capacity** of \(G\) is defined as

\[
\lim_{t \to \infty} \frac{1}{t} \log \alpha (G^t)
\]

By (1), \(\alpha(G^{s+t}) \geq \alpha(G^s)\alpha(G^t)\)

\[
\log \alpha (G^{s+t}) \geq \log \alpha (G^s) + \log \alpha (G^t)
\]

By Fekete’s lemma \(\lim_{t \to \infty} \frac{1}{t} \log \alpha (G^t)\) exists and is equal to \(\sup_{t \geq 1} \frac{1}{t} \log \alpha (G^t)\).
If $\alpha(G) = \gamma(G)$, then

$$(\gamma(G))^t \geq \gamma(G^t) \geq \alpha(G^t) \geq (\alpha(G))^t$$

and so equality holds throughout. Thus the Shannon capacity of $G$ is $\log \alpha(G)$.

What are the minimal graphs that do not satisfy $\alpha(G) = \gamma(G)$? Those are precisely minimally imperfect graphs.

**Theorem** (Lovász) The Shannon capacity of $C_5$ is $(\log 5)/2$.

We do not know the Shannon capacity of $C_7$ or other odd holes or odd antiholes.
Let $\gamma(G) := \chi(G^c) = \min \# \text{ of cliques covering the vertices of } G$.

**Observations:**
1. $\alpha(G_1 \boxtimes G_2) \geq \alpha(G_1)\alpha(G_2)$
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**Proof of (3).** Let $K_1, \ldots, K_r$ be a cover by cliques of $G_1$.

Let $L_1, \ldots, L_s$ be a cover by cliques of $G_2$.

Then $\{K_i \times L_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ is a cover of $G_1 \boxtimes G_2$ by $rs$ cliques, as desired.

Let $G^t := G \boxtimes G \boxtimes \cdots \boxtimes G$ ($t$ times). The **Shannon capacity** of $G$ is defined as

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By (1), $\alpha(G^{s+t}) \geq \alpha(G^s)\alpha(G^t)$

$$\log \alpha(G^{s+t}) \geq \log \alpha(G^s) + \log \alpha(G^t)$$

By Fekete’s lemma $\lim_{t \to \infty} \frac{1}{t} \log \alpha(G^t)$ exists and is equal to $\sup_{t \geq 1} \frac{1}{t} \log \alpha(G^t)$. 

38