Extremal problems

How many edges can a triangle-free graph on $n$ vertices have?

$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \approx \frac{n^2}{4}$ edges

**Theorem** (Matel) If $G$ has no triangle and $n = |V(G)|$, then $|E(G)| \leq \frac{n^2}{4}$.

**Proof.** Let $uv \in E(G)$.

Neighbors disjoint $\Rightarrow$

$\deg(u) + \deg(v) \leq n$

$$n|E(G)| \geq \sum_{uv \in E(G)} (\deg(u) + \deg(v)) =$$

$$= \sum_{w \in V(G)} \deg^2(w) \geq \frac{1}{n} \left( \sum_{w \in V(G)} \deg(w) \right)^2 = \frac{4}{n} |E(G)|^2$$

$\Rightarrow |E(G)| \leq \frac{n^2}{4}$
Fix $r$. What is the maximum number of edges a graph with no $K_r$ subgraph can have?

An extremal graph $T_{r-1}(n)$, “the Turán graph”, is the complete multipartite graph with $r - 1$ parts that are as close as possible to each other in size. It has all edges between different parts. Each part has size $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$.

If $r - 1$ divides $n$, then

$$|E(T_{r-1}(n))| = \binom{r-1}{2} \left( \frac{n}{r-1} \right)^2 = \frac{(r-1)(r-2)}{2} \frac{n^2}{(r-1)^2} = \frac{1}{2} \frac{r-2}{r-1} n^2$$
**Theorem** (Turán) If $G$ is a graph on $n$ vertices with no $K_r$-subgraph, then

$$|E(G)| \leq |E(T_{r-1}(n))|$$

with equality if and only if $G$ is isomorphic to $T_{r-1}(n)$.

**Proof #1.** Let $G$ have no $K_r$-subgraph and maximum number of edges. We claim $G$ is complete multipartite. If not, then $\sim$ is not transitive, and so there exist $x, y, z$:

![Graph](image)

If $\deg(x) > \deg(y)$, then deleting $y$ and cloning $x$ produces a graph with no $K_r$-subgraph and $>|E(G)|$ edges, a contradiction. So WMA $\deg(x) \leq \deg(y)$ and similarly $\deg(z) \leq \deg(y)$. Now deleting both $x$ and $z$ and cloning $y$ twice produces a graph with no $K_r$-subgraph and $>|E(G)|$ edges, a contradiction. This proves our claim that $G$ is complete multipartite.

The maximality of $G$ implies that $G$ is isomorphic to $T_k(n)$ for some $k$. Since $G$ has no $K_r$-subgraph it follows that $k \leq r - 1$, and by comparing the degrees of vertices in $T_k(n)$ and $T_{r-1}(n)$ we conclude that the maximality of $G$ implies that $k = r - 1$. 

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Lemma.

\[ \alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}, \]

with equality if and only if \( G \) is a disjoint union of cliques.

Proof. Let \(<\) be a linear ordering of \( V(G) \).

\[ I(<) := \{ v \in V(G): \forall w \quad vw \in E(G) \Rightarrow v < w \} \]

Then \( I(<) \) is independent. Now choose \(<\) uniformly at random.

\[ X(<) := |I(<)| = \sum_{v \in V(G)} 1_{[v \in I]} \]

\[ EX = \sum_{v \in V(G)} E 1_{[v \in I]} = \sum_{v \in V(G)} P[v \in I] = \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \]

There exists \(<\) such that \(|I(<)| \geq EX; then \( I(<) \) is an independent set of size \( \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \), as required.

Assume now \( G \) is not a union of cliques. We will prove inequality is strict. \( \Rightarrow \exists \ x, y, z: \)

![Diagram](attachment:image.png)
Define

\[<_1 : \ x, y, z, \ldots \ldots\]

\[<_2 : \ x, z, y, \ldots \ldots\]

Then \(I(<_1) = \{x, \ldots \} \) and \(I(<_2) = \{x, z, \ldots \}\), and so \(|I(<_1)| < |I(<_2)|\). Thus \(X\) is not constant, and hence there exists a linear ordering \(<\) such that \(X(<) > EX\), and so

\[\alpha(G) > \sum_{v \in V(G)} \frac{1}{\deg(v) + 1},\] as desired. \(\square\)

**Corollary.** If \(G\) has \(n\) vertices and \(e\) edges, then

\[\alpha(G) \geq \frac{n^2}{2e + n}\]

**Proof.** If \(a_1 + \cdots + a_n = \text{const}\), then \(\sum_{i=1}^{n} \frac{1}{a_i}\) is minimized when the \(a_i\)'s are equal. Thus

\[\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \geq \sum_{v \in V(G)} \frac{1}{2e + 1} = \frac{n}{2e + 1} = \frac{n^2}{2e + n}\] \(\square\)
Theorem 2. Let $H$ be a graph on $n$ vertices such that $|E(H)| = E(T^c_{r-1}(n))$. Then $\alpha(H) \geq r - 1$ with equality if and only if $H \cong T^c_{r-1}(n)$.

Proof of Turán’s theorem, assuming Theorem 2. Let $G$ have $n$ vertices and no $K_r$ subgraph. WMA

$$|E(G)| \geq |E(T_{r-1}(n))|,$$

for otherwise we are done. Let $H$ be a spanning supergraph of $G^c$ with $|E(T^c_{r-1}(n))|$ edges. Then $\alpha(H) \leq r - 1$. By Theorem 2 $\alpha(H) \geq r - 1$, and so $H \cong T^c_{r-1}(n)$. Thus $G$ has a spanning subgraph isomorphic to $T_{r-1}(n)$, and hence $G \cong T_{r-1}(n)$, because adding any edge to $T_{r-1}(n)$ creates a $K_r$-subgraph. \hfill \Box

Proof of Theorem 2. By the lemma

$$\alpha(H) \geq \sum_{v \in V(H)} \frac{1}{\deg_H(v) + 1} \geq \sum_{v \in T^c_{r-1}(n)} \frac{1}{\deg_{T^c_{r-1}}(v) + 1} = r - 1$$

The second inequality holds because the previous expression is minimized when the degrees are as close to each other as possible.

Equality in 1st inequality $\iff H$ is a union of cliques.

Equality in 2nd inequality $\iff$ degrees are as close to each other as possible.

$\Rightarrow$ Equality in Theorem 2 if and only $H \cong T^c_{r-1}(n)$. \hfill \Box