EXTRA CREDIT PROBLEM #1

Consider a 10 × 10 board. At the beginning each square is white. We make a series of the following moves. In each move, we paint black the four squares made by the intersection of a pair of rows and a pair of columns. A move can be made if at least one of the four squares is white. What is the maximum number of moves that can be made?

Solution (Shijie Xie). Let $G$ be the bipartite graph with vertices the rows and columns of the board. Thus the squares of the board are in 1-1 correspondence with edges of $G$. Let

$N_i =$ number of squares (edges) painted black during move $i$
$c_i =$ number of components of the subgraph of $G$ consisting of edges that were black before move $i$

$T_i = c_i - c_{i+1}$

Thus $N_1 = 4$, $c_1 = 20$, $c_2 = 17$, $T_1 = 3$.

Claim. $T_i \leq N_i - 1$ for every $i$.

The claim implies that after $n$ moves

$$100 \geq N_1 + \cdots + N_n \geq T_1 + \cdots + T_n + n \geq 19 + n$$

and so $n \leq 81$.

To show that 81 moves are possible consider the moves that use rows $i$ and 10 and columns $j$ and 10 for all $i, j = 1, \ldots, 9$. 
Random graphs

Let $0 < p < 1$, it may depend on $n$. Let $G(n, p)$ be the probability space of all graphs $G$ with $V(G) = \{1, 2, \ldots, n\}$, where $G$ has probability

$$p^{|E(G)|}(1 - p)^{\binom{n}{2} - |E(G)|}$$

Edges exist with probability $p$, independently of each other.

Definition. We say that a.e. graph in $G(n, p)$ has property $\Pi$ if

$$\lim_{n \to \infty} P[G \text{ has } \Pi] = 1$$

Theorem. Let $0 < p < 1$ be fixed. Let $k, \ell$ be fixed. Then a.e. graph $G$ in $G(n, p)$ has the property that for all distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_\ell$ there exists a vertex $v \in V(G) - \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_\ell\}$ such that $v$ is adjacent to $x_1, x_2, \ldots, x_k$ and not adjacent to $y_1, y_2, \ldots, y_\ell$.

Proof. For fixed $x_1, \ldots, x_k, y_1, \ldots, y_\ell$ let us say that $v$ works if $v \in V(G) - \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$, and $v$ is adjacent to $x_1, \ldots, x_k$ and not adjacent to $y_1, \ldots, y_\ell$.

The probability that a given $v$ works is $p^k(1 - p)^\ell$.

The probability it does not work is $1 - p^k(1 - p)^\ell$.

The probability that no $v$ works is $(1 - p^k(1 - p)^\ell)^{n-k-\ell}$.
The probability that $\exists x_1, \ldots, x_k, y_1, \ldots, y_\ell$ such that no $v$ works is

$$\leq n^{k+\ell}(1 - p^k(1 - p)\ell)n^{-k-\ell} \rightarrow 0$$
as $n \rightarrow \infty$. $\square$

**Consequences.** When $p$ is fixed:

1. a.e. graph in $G(n, p)$ has diameter $\leq 2$
2. for fixed $k$ a.e. graph in $G(n, p)$ is $k$-connected.

**Proof.** If $G$ is not $k$-connected, then $\exists a, b, y_1, \ldots, y_\ell$ where $\ell < k$, such that $\not\exists a - b$ path in $G\{y_1, y_2, \ldots, y_\ell\}$. Apply theorem to $a, b, y_1, y_2, \ldots, y_\ell$. $\exists v$ adjacent to $a, b$, not equal to $a, b, y_1, \ldots, y_\ell$, a contradiction.

3. For every fixed graph $H$, a.e. graph in $G(n, p)$ has an induced subgraph isomorphic to $H$.

**Proof.** Pick $v \in V(H)$. By induction a.e. graph in $G(n, p)$ has an induced subgraph isomorphic to $H \setminus v$. Let $x_1, \ldots, x_k$ be the vertices of $H \setminus v$ adjacent to $v$ and let $y_1, \ldots, y_\ell$ be the vertices of $H \setminus v$ not adjacent to $v$. Apply the theorem.
Markov’s inequality. Let $X$ be a non-negative random variable on a probability space with $0 < EX < \infty$. Then for all $t > 0$

$$P[X \geq tEX] \leq \frac{1}{t}$$

Proof. 

$$EX = \int X \, dP \geq \int_{[X \geq tEX]} X \, dP \geq tEX \int_{[X \geq tEX]} 1 \, dP =$$

$$= tEX \cdot P[X \geq tEX]$$

and so $P[X \geq tEX] \leq \frac{1}{t}$.

Corollary. $P[X \geq z] \leq \frac{EX}{z}$ for every $z > 0$.

Proof. Let $z = tEX$. 


**Theorem** (Erdös 1959) For every two integers $k, l$ there exists a graph $G$ with $\chi(G) \geq k$ and no cycles of length at most $l$.

**Proof.** We will consider $G(n, p)$, where $p = p(n)$ will be determined later. We first prove that for a suitable choice of $p$ a.e. graph in $G(n, p)$ has few short cycles. Let $X_i(G)$ be the number of cycles in $G$ of length exactly $i$. Then

$$X(G) := \sum_{i=3}^{l} X_i(G)$$

is the number of cycles in $G$ of length at most $l$. Now

$$EX = \sum_{i=3}^{l} EX_i = \sum_{i=3}^{l} \sum_{|A|=i} \sum_{\sigma} P[\sigma \text{ determines a cycle}] =$$

$$= \sum_{i=3}^{l} \binom{n}{i} \frac{1}{2} (i - 1)! p^i \leq \frac{1}{2} \sum_{i=3}^{l} \frac{n^i}{i!} (i - 1)! p^i \leq \sum_{i=3}^{l} (np)^i$$

Now it seems sensible to choose $p = p(n)$ so that $np = n^\theta$ for some $\theta > 0$. Thus the above is equal to

$$\sum_{i=3}^{l} n^{\theta i} \leq ln^{\theta l}$$
for $n$ sufficiently large. If we choose $\theta < 1/l$, then $EX = o(n)$. By Markov's inequality

$$P[X \geq n/2] \leq \frac{2EX}{n} = o(1)$$

and so for all sufficiently large $n$ we have $P[X \geq n/2] < 1/2$. That is,

(1) the probability that $G \in \mathcal{G}(n, p)$ has $\geq n/2$ cycles of length $\leq l$ is strictly less than $1/2$

Next we give a lower bound on $\chi(G)$. For that we will use the inequality $\chi(G)\alpha(G) \geq n$. So we need an upper bound on $\alpha(G)$ and so we need an upper bound on $P[\alpha(G) \geq t]$.

$$P[\alpha(G) \geq t] = P[G \text{ has an independent set of size } t] \leq$$

$$\sum_A P[A \text{ is an independent set in } G] = \binom{n}{t} (1 - p)^{\binom{t}{2}} <$$

$$< n^t (1 - p)^{\binom{t}{2}} \leq n^t e^{-p \binom{t}{2}} = \left[ne^{-p(t-1)/2}\right]^t$$

where the second inequality uses $1 + x \leq e^x$.

If $t - 1 = \frac{3}{p} \log n$, then

$$[ne^{-p(t-1)/2}]^t = \left[n \cdot n^{-3/2}\right]^t = o(1)$$

Thus for $t = \frac{3}{p} \log n$ and sufficiently large $n$
(2) \( P[\alpha(G) \geq t] < 1/2 \)

By (1) and (2) there exists a graph on \( n \) vertices with \( \leq n/2 \) cycles of length \( \leq l \) and \( \alpha(G) \leq t \). By deleting one vertex from each cycle of length \( \leq l \) we arrive at an induced subgraph \( G' \) on at least \( n/2 \) vertices with no cycle of length \( \leq l \) and \( \alpha(G') \leq t \). Now

\[
\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{t} \geq \frac{3}{p} \log n = \frac{n}{6n^{1-\theta} \log n} = \frac{n^\theta}{6 \log n} \to \infty
\]

and so for \( n \) sufficiently large we have \( \chi(G') \geq k \), as desired.