Application to geometry

\(cr(G) := \text{minimum number of crossings in a drawing of } G \text{ in the plane (in which crossings are allowed). More precisely, our “drawings” now allow edges to intersect, but}

1. \(|e \cap e'| \text{ is finite for distinct } e, e' \in E(G)\)
2. each point of \(\mathbb{R}^2\) belongs to \(\leq 2\) edges.

So the number of crossings in a drawing is \(\sum_{\{e,e'\}} |e \cap e'|\), and \(cr(G)\) is the minimum, over all drawings \(\Gamma\) of \(G\), of the number of crossings in \(\Gamma\).

Examples. \(cr(K_5) = cr(K_{3,3}) = 1, cr(K_6) = 3\) (exercise)

Fact. Computing \(cr(G)\) is NP-hard.

Conjecture. \(cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor\)

Remark. Let \(cr'(G)\) denote the minimum, over all drawings \(\Gamma\) of \(G\), of the number of unordered pairs of edges that cross in \(\Gamma\). It is not known whether \(cr(G) = cr'(G)\) for all graphs \(G\).

Lemma. \(cr(G) \geq |E(G)| - 3|V(G)|\).

Proof. If not, then remove \(cr(G)\) edges, one from each crossing, to get a planar graph on \(n\) vertices and \(\geq |E(G)| - cr(G) > 3n\) edges, a contradiction.
Crossing Number Lemma. (Ajtai, Chvatal, Newborn, Szemerédi; Leighton) Let $G$ be a simple graph. Then

$$cr(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2} - |V(G)|$$

Proof. (Székely) Let $c$ be the crossing number of $G$, let $n = |V(G)|$, $m = |E(G)|$. WMA $m \geq 4n$, for o.w. RHS is negative. Let $p \in (0,1)$, TBD. Choose a random subset $V \subseteq V(G)$ by picking each vertex independently at random with probability $p$. The expected number of vertices is $pn$
edges is $p^2m$
crossings is $p^4c$

By the lemma

$$p^4c \geq p^2m - 3pn$$

and so

$$c \geq \frac{m}{p^2} - 3 \frac{n}{p^3}$$

Choose $p = \frac{4n}{m}$ (which is $< 1$, because $m \geq 4n$) to get

$$c \geq \frac{m^3}{64n^2}$$

□
Is the expected number of crossings really $p^4c$?
Let $I(n, m)$ be the maximum number of possible incidences between $n$ points and $m$ lines in the plane. That is,

$$I(n, m) = \max \left| \{(p, L) : p \in P, L \in \mathcal{L}, p \in L\} \right|$$

where the maximum is taken over all sets $P \subseteq \mathbb{R}^2$ and sets of lines $\mathcal{L}$ such that $|P| = n$ and $|\mathcal{L}| = m$.

**Example.** $I(3, 3) \geq 6$

![Diagram showing incidences](image)

**Theorem.** (Szemerédi-Trotter) For all $m, n \geq 1$,

$$I(n, m) = O(n^{2/3}m^{2/3} + n + m)$$

and the bound is asymptotically tight.

**Proof.** (Szekély) Let $P, \mathcal{L}$ be a system of points and lines realizing $I(n, m)$. Define a topological graph (= graph drawn with crossings) $G$ by $V(G) = P$ and $E(G) = \text{subsets of lines in } \mathcal{L}$ connecting consecutive points.
A line $L \in \mathcal{L}$ containing $k$ points contributes $k - 1$ edges. So

$$I(n, m) = \sum_{L \in \mathcal{L}} \# \text{ of points on } L =$$

$$= \sum_{L \in \mathcal{L}} (1 + \# \text{ edges of } G \text{ contributed by } L) = |E(G)| + m.$$ 

By the Crossing Number Lemma

$$\frac{1}{2} m^2 \geq \binom{m}{2} \geq cr(G) \geq \frac{1}{64} \frac{|E(G)|^3}{n^2} - n$$

$$|E(G)|^3 \leq 32m^2n^2 + 64n^3$$

$$I(n, m) = |E(G)| + m \leq O\left(\frac{m^{2/3}n^{2/3}}{n^2} + n + m\right)$$