What is $\det(M + E_{\nu\nu})$? WMA $\nu = 1$.

\[
det(M + E_{11}) = \sum_{\sigma} \text{sgn} (\sigma) \prod_{i=1}^{n} (M + E_{11})_{i\sigma(i)} = \\
= \sum_{\sigma: \sigma(1) = 1} \text{sgn} (\sigma) (M_{11} + 1) \prod_{i=2}^{n} M_{i\sigma(i)} + \\
+ \sum_{\sigma: \sigma(1) \neq 1} \text{sgn} (\sigma) \prod_{i=1}^{n} M_{i\sigma(i)} = \\
= \sum_{\sigma} \text{sgn} (\sigma) \prod_{i=1}^{n} M_{i\sigma(i)} + \sum_{\sigma: \sigma(1) = 1} \text{sgn}(\sigma) \prod_{i=2}^{n} M_{i\sigma(i)} = \\
= \det(M) + \det M(1)
\]
**Kirchhoff’s Matrix Tree Theorem.** Let $G$ be a multigraph, let $L$ be its Laplacian matrix, let $k \in \{1,2, \ldots, k\}$, and let $L(k)$ denote the matrix obtained from $L$ by deleting the $k^{th}$ row and $k^{th}$ column. Then $\tau(G) = \det L(k)$.

**Proof.** If $G$ is disconnected, then $\tau(G) = 0 = \det L(k)$.

WMA $G$ is connected and loopless.

If $|E(G)| = 0$, then $\tau(G) = 1 = \det L(k)$.

We proceed by induction on $|E(G)|$.

Recall

$$\tau(G) = \tau(G \setminus e) + \tau(G/e)$$

Enough to show

$$(*) \quad \det L_G (u) = \det L_G \setminus e (u) + \det L_G/e (w)$$

where $e = uv$ and $w$ is the new vertex of $G/e$

$$L_G(u) = L_G \setminus e (u) + E_{vv}$$

$$\det L_G (u) = \det[L_G \setminus e (u) + E_{vv}]$$

$$= \det L_G \setminus e (u) + \det L_G \setminus e (u,v)$$

$$= \det L_G \setminus e (u) + \det L_G/e (w)$$

This proves $(*)$, and hence the theorem. \qed
A directed multigraph is a triple \((V, E, \psi)\), where \(V, E\) are finite sets and \(\psi\) is an incidence relation that assigns to every edge \(e \in E\) an ordered pair of not necessarily distinct vertices of \(V\), called its ends.

We denote the outdegree of a vertex \(v\) by \(\text{deg}^+(v)\) and the indegree by \(\text{deg}^-(v)\).