Food for thought.

Suppose we have 3 internally disjoint s-t paths in $G$.

Suppose also $\not\exists$ a set $X \subseteq V(G) - \{s, t\}$ of size 3 or less such that $G \setminus X$ has no s-t path. Then, by Menger’s theorem, there exist 4 internally disjoint paths.

Where are they?
Block structure of (connected) graphs.

A **cut-vertex** in a multigraph $G$ is a vertex $v$ such that $E(G)$ can be partitioned into disjoint non-empty sets $E_1, E_2$ such that $v$ is the only vertex such that both $E_1$ and $E_2$ include an edge incident with $v$.

**Example 1.**

![Diagram](image1)

**Example 2.** $G \setminus v$ disconnected.

![Diagram](image2)

**Definition.** A **block** is a connected multigraph with no cut-vertex.
Examples.

Every loopless and 2-connected multigraph.

**Definition.** A block of a multigraph $G$ is a maximal submultigraph that is a block.

**Proposition.** (i) Every multigraph is a union of its blocks.

(ii) If $B_1, B_2$ are distinct blocks of $G$, then $|V(B_1) \cap V(B_2)| \leq 1$, and if $v \in V(B_1) \cap V(B_2)$, then $v$ is a cut-vertex.

**Proof.** (i) Immediate.

(ii) Suppose $x, y \in V(B_1) \cap V(B_2)$, $x \neq y$

$\Rightarrow B_1 \cup B_2$ is block
If $B_1, B_2$ are 2-connected $\Rightarrow B_1 \cup B_2$ 2-connected.

**Proof.** Let $z \in V(B_1 \cup B_2)$. Then $B_1 \setminus z, B_2 \setminus z$ are connected, and they intersect $\Rightarrow (B_1 \setminus z) \cup (B_2 \setminus z) = (B_1 \cup B_2) \setminus z$ is connected.
Given a multigraph, define a graph $F$ as follows: $V(F) = \mathcal{B} \cup \mathcal{C}$.

$\mathcal{B} = \text{all blocks of } G; \mathcal{C} = \text{all cut-vertices of } G.$

$B \in \mathcal{B}$ is adjacent to $c \in \mathcal{C}$ if $c \in V(B)$.

**Theorem.** $F$ is a forest, and if $G$ is connected, then it is a tree.

**Definition.** This is called the **block structure of a graph**.

**Proof.** Exercise.
Given a multigraph, define a graph $F$ as follows: $V(F) = \mathcal{B} \cup \mathcal{C}$

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**Proof.** Exercise.
**Theorem.** (Ear structure of 2-connected graphs).
Let $G$ be a 2-connected graph. Then $G$ can be written as $G = G_0 \cup G_1 \cup \cdots \cup G_k$, where

(i) $G_0$ is a cycle, and

(ii) for $i = 1, 2, \ldots, k$, $G_i$ is a path with both ends in $G_0 \cup \cdots \cup G_{i-1}$, and otherwise disjoint from it.

**Proof.** $\exists$ cycle. We can pick $G_0, G_1, \ldots, G_k$ satisfying (i) and (ii) with $k$ maximum.
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A lemma about 3-connected graphs

**Theorem.** Let $G$ be a 3-connected graph on $\geq 5$ vertices. Then $G$ has an edge $e$ such that $G/e$ is 3-connected.

**Proof.** Suppose not. Thus $\forall e = uv \exists x \in V(G)$ such that $G\{u, v, x\}$ is disconnected. Pick $e = uv$ and $x$ as above and a component $K$ of $G\{u, v, x\}$ such that $|V(K)|$ is maximum. Note $u$ has a nbr in $K$, for o.w. $G\{v, x\}$ is disconnected. Same for $v, x$. Same for other components. Let $y$ be a nbr of $x$ not in $V(K) \cup \{u, v\}$. 
For the edge $e' = xy \ni z \in V(G)$ such that $G\{x, y, z\}$ is disconnected. Let $K'$ be the subgraph induced by $V(K) \cup \{u, v\}$.

**Claim.** $K'\setminus z$ is connected.

**Proof.** Let $a, b \in V(K') - \{z\}$. We must show that there exists an $a$- $b$ path in $K'\setminus z$. Since $G$ is 3-connected there exists an $a$-$b$ path $P$ in $G\{z, x\}$. If $P$ is a path in $K'\setminus z$, then we are done, and so we may assume not. Thus $P$ leaves $K'$ through $u$ or $v$ and re-enters through $v$ or $u$. In either case it uses both $u$ and $v$. Let $P'$ be obtained from $P$ by short cutting using the edge $e = uv$. Then $P'$ is an $a$-$b$ path in $K'\setminus z$, as desired. This proves the claim.

By the claim, $K'\setminus z$ is a subgraph of a component $K''$ of $G\{x, y, z\}$. But $|V(K'')| \geq |V(K')| - 1 \geq |V(K)| + 1$ contrary to the choice of $u, v, x, K$. □