Matchings in bipartite graphs

Let $G$ be a bipartite graph with bipartition $(A, B)$. A matching $M$ is a complete matching from $A$ to $B$ if it saturates every vertex of $A$. If $|A| = |B|$, then a complete matching from $A$ to $B$ is the same as a perfect matching.

**Obstruction:**

$N(S) := \{v \notin S: v$ is adjacent to a vertex in $S\}$

If $|N(S)| < |S|$ for some $S \subseteq A$, then $\nexists$ complete matching $A$ to $B$.

**Theorem.** (Hall) A bipartite graph with bipartition $(A, B)$ has a complete matching from $A$ to $B$ if and only if $|N(S)| \geq |S|$ for every $S \subseteq A$. 

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**Proof #1.** Using Menger’s theorem

\(\Rightarrow\) already done

\(\Leftarrow\) If there exist \(|A|\) disjoint paths from \(A\) to \(B\), then their edge-sets form a complete matching from \(A\) to \(B\). Thus WMA \(\not\exists |A|\) disjoint \(A\)-\(B\) paths. By Menger’s theorem \(\exists X \subseteq V(G)\) such that \(G \setminus X\) has no \(A\)-\(B\) path and \(|X| < |A|\).

Let \(S := A - X\). Then \(N(S) \subseteq X \cap B\) and hence

\[|N(S)| \leq |X \cap B| = |X| - |X \cap A| < |A| - |X \cap A| = |S|,\]

a contradiction. \(\square\)
Proof #2. From first principles (⇐ only)

Case 1. $|N(S)| > |S|$ for every $\emptyset \neq S \subsetneq A$.

Pick $v \in A$ and a neighbor $u$ of $v$.

Apply induction to $G\setminus\{u, v\}$.

Case 2. $|N(S)| = |S|$ for some $\emptyset \neq S \subsetneq A$.

Let $G_1 := G[S \cup N(S)]$

$G_2 := G\setminus(S \cup N(S))$

Apply induction to $G_1$ and $G_2$.

$G_1$ clearly satisfies the induction hypothesis.

To see that $G_2$ satisfies the induction hypothesis for $L \subseteq A - S$, look at $N_G(L \cup S)$.

$$|N_G(S)| + |N_{G_2}(L)| = |N_G(S \cup L)| \geq |S \cup L| = |S| + |L|$$

and so $|N_{G_2}(L)| \geq |L|$, as desired. \qed
Perfect matchings in (not necessarily bipartite) graphs

An obstruction:

Let $o(H) := \# \text{ of odd components of } H$.

If $o(G \setminus X) > |X|$ for some $X$, then $G$ has no perfect matching.

**Tutte’s 1-factor theorem** (1947). A graph $G$ has a perfect matching if and only if $o(G \setminus X) \leq |X|$ for every $X \subseteq V(G)$. 
**Definition.** Let $M$ be a matching in $G$. A cycle $C$ in $G$ of length $2k + 1$ containing $k$ edges of $M$ is called an $M$-blossom. Let $G/C$ denote the graph obtained from $G$ by contracting all edges of $C$ and deleting all loops and parallel edges.

**Lemma.** Let $M$ be a matching in $G$, and let $C$ be an $M$-blossom in $G$. Let $G' := G/C$ and $M' := M - E(C)$. If $M$ is a maximum matching in $G$, then $M'$ is a maximum matching in $G'$.

**Proof.** Suppose not. Then $\exists M'$-augmenting path $P'$ in $G'$. We will exhibit an $M$-augmenting path in $G$. Let $w$ be the new vertex of $G'$. WMA $w \in V(P')$, for otherwise $P'$ is as desired.

The vertex $w$ divides $P'$ into $P_1$ and $P_2$. Let $u, v$ be the ends of $P'$. WMA by symmetry that the edge of $P_2$ incident with $w$ is in $M$. 
Then in $G$ the path $P_2$ becomes a path from $v$ to the tip of the blossom, and $P_1$ becomes a path from $u$ to the blossom. Follow $P_1$ from $u$ to $u' \in V(C)$, then follow $C$ along the even path from $u'$ to the tip, and then follow $P_2$. That gives an $M$-augmenting path in $G$. □
Definition. Let $M$ be a matching in $G$, and let $r \in V(G)$ be $M$-unsaturated. An $M$-alternating tree rooted at $r$ is a tree $T$ such that

(i) $T$ is a subgraph of $G$
(ii) $r \in V(T)$
(iii) every path in $T$ with end $r$ is $M$-alternating
(iv) if $e \in M$ is incident with a vertex of $T$, then $e \in E(T)$
Given an $M$-alternating tree $T$ let
\[
A(T) = \{ v \in V(T) : v \text{ is at odd distance from } r \text{ in } T \}
\]
\[
B(T) = \{ v \in V(T) : v \text{ is at even distance from } r \text{ in } T \}
\]

**Theorem.** Let $G$ be a graph, let $M$ be a maximum matching in $G$, let $r \in V(G)$ be $M$-unsaturated, and let $T$ be an $M$-alternating tree rooted at $r$. Then there exists a set $X \subseteq V(G)$ such that $X \cap V(T) \subseteq A(T)$ and $o(G \setminus X) > |X|$.

Note that this implies Tutte’s theorem.