The Pfaffian of a skew symmetric matrix $A$

$$
Pf(A) = \sum \text{sign} \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix} a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n}
$$

the summation over all partitions $\{\{i_1,j_1\},\{i_2,j_2\},\ldots,\{i_n,j_n\}\}$ of $[2n]$ into unordered pairs.

Lemma $Pf^2(A) = \det(A)$
In particular, $Pf(A)$ can be computed efficiently.
The Pfaffian of a skew symmetric matrix $A$

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the summation over all partitions $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$ of $[2n]$ into unordered pairs.

Now let $A$ be a skew adjacency matrix of a graph $G$.

Order the pairs $(i_k, j_k)$ to make sure $a_{i_kj_k} = 1$. Then

$$\text{Pf}(A) = \sum \text{sgn} \begin{pmatrix} 1 & 2 & \cdots & 2n - 1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix} \text{sgn}_D(M)$$
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DEF An orientation \( D \) of a graph \( G \) is Pfaffian if 
\( sgn_D(M) = sgn_D(M') \) for every two perfect matchings \( M, M' \).

In that case the number of perfect matchings can be efficiently calculated.
\[ \text{Pf}(A) = \sum \text{sgn} \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix} \]

**DEF** An orientation \( D \) of a graph \( G \) is Pfaffian if \( \text{sgn}_D(M) = \text{sgn}_D(M') \) for every two perfect matchings \( M, M' \).

**Equivalently:** Every even cycle \( C \) such that \( G \setminus V(C) \) has a perfect matching ("central cycle") has an odd number of edges directed in either direction (is "oddly oriented").

**Equivalently:** Either \( G \) has no perfect matching, or for some perfect matching \( M \), every \( M \)-alternating cycle is oddly oriented.
Example:
Example:
Example:

Oddly oriented
Example:

Not oddly oriented
The number of perfect matchings is $\eta^{N(1+o(1))}$. 
THM (Kasteleyn 1963) Every planar graph has a Pfaffian orientation.

PROOF Orient $G$ so that $\forall$ cycle $C$: $C$ is clockwise odd $\iff C$ encloses even number of vertices of $G$. 