INTRODUCTION TO PLANAR GRAPHS

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ABSTRACT

The topic of planar graphs is covered in many books and articles, but the treatment usually relies on intuition or on deep topological theorems that are quoted without proof. I give a self-contained rigorous introduction to planar graphs.

1 The Jordan curve theorem and Euler's formula

Throughout these notes *graphs* are allowed to have loops and multiple edges.

Definition 1.1. A *polygonal arc* is a set $A \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the interval $[0,1]$. The images of 0 and 1 under the homeomorphism are called the *ends* of A. A *polygon* is a set $B \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. By a *bend* of a polygonal arc or a polygon P we mean a point of P where two different straight line segments meet. Thus P has finitely many bends.

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and for $x, y \in \Omega$ let us define $x \sim y$ if there exists a polygonal arc $A \subseteq \Omega$ with ends x and y. Then $\sim$ is an equivalence relation, and the equivalence classes are called the *arcwise connected components of* Ω. If $x \sim y$ for any two $x, y \in \Omega$, then we say that Ω is *arcwise connected*. Now if $F \subseteq \mathbb{R}^2$ is closed, we say that an arcwise connected component of $\mathbb{R}^2 - F$ is a *face* of F.

Theorem 1.3 (Jordan curve theorem for polygons). *Every polygon P has exactly two faces, of which exactly one is bounded. The boundary of each of the two faces is P.*

Proof. For $x \in \mathbb{R}^2 - P$ and a half-line L originating in x and containing no bends of P let $\pi(x, L)$ denote the number of intersections of L with P modulo 2. It is easy to check that this can be extended to all half-lines originating in x in such a way that $\pi(x, L)$ does not depend on L. Let us call that value $\pi(x)$. Furthermore, it follows that the function π is continuous, and hence is constant on each arcwise connected component of $\mathbb{R}^2 - P$. By choosing two points x_1 and x_2 close

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to each other, but on opposite sides of a straight-line segment of P we can show that $\pi(x_1) \neq \pi(x_2)$. Thus P has at least two faces.

Suppose now that P has at least three faces, and choose a point in each, say x_1, x_2, x_3. Pick a point x on the boundary of P inside a straight-line segment S of P. Pick a small open neighborhood O of x such that $O \cap P = O \cap S$. By shooting a half-line from each of x_1, x_2, x_3 toward P and then following the boundary of P until we hit O we see that each of x_1, x_2, x_3 can be reached from a point in O by a polygonal arc not intersecting P. But $O - P$ has at most two arcwise connected components, and hence P has at most two faces. It follows that every point of $O \cap S$ belongs to the boundary of both faces of P, and since x was arbitrary, we deduce that the boundary of both faces of P is P. We leave it as an exercise to show that exactly one of the faces is bounded. ☐

Definition 1.4. A *plane graph* is graph G such that

(i) $V(G) \subseteq \mathbb{R}^2$,

(ii) for every non-loop edge $e \in E(G)$ with ends u and v there exists a polygonal arc A with ends u and v such that $e = A - \{u, v\} \subseteq \mathbb{R}^2 - V(G)$,

(iii) for every loop $e \in E(G)$ incident with $u \in V(G)$ there exists a polygon P containing u such that $e = P - \{u\} \subseteq \mathbb{R}^2 - V(G)$, and

(iv) if $e, e' \in E(G)$ are distinct, then $e \cap e' = \emptyset$.

Thus an edge of a plane graph is a subset of the plane that includes no vertices, not even its ends. We define the *point set of G* to be the set $\bigcup_{e \in E(G)} e \cup V(G)$, and by abusing notation we shall denote this set also by G. Thus the faces of G are the arcwise connected components of $\mathbb{R}^2 - G$. The set of faces of a plane graph G will be denoted by $F(G)$.

Definition 1.5. A graph G is *planar* if it is isomorphic to a plane graph Γ. We say that Γ is a (planar) drawing of G.

Lemma 1.6. *Let G be a plane graph, let $e \in E(G)$, and let $x_1, x_2 \in \mathbb{R}^2 - G$ be two points such that the straight line segment connecting them intersects e exactly once, and is otherwise disjoint from G. Let f_i be the face of G that includes x_i. Then e is a subset of the boundary of both f_1 and f_2 and is disjoint from the boundary of every other face of G.*

Proof. This follows by a similar argument as the second half of the proof of Theorem 1.3. ☐

Corollary 1.7. *If G is a plane graph and $f \in F(G)$, then the boundary of f is the point set of a subgraph of G. In particular, if the boundary of f includes a point belonging to an edge $e \in E(G)$, then it includes the entire edge e.*

Definition 1.8. Let G, e, f_1, f_2 be as in Lemma 1.6. We will refer to f_1 and f_2 as the two faces *incident* with e. Please note that f_1 and f_2 need not be distinct.

Lemma 1.9. *Let G be a plane graph that is a forest. Then G has exactly one face.*

Proof. Exercise. Use induction on the number of vertices plus the number of bends. ☐

Lemma 1.10. *Let G' be a subgraph of a plane graph G. Then*

(i) *every face of G is a subset of a face of G',*

(ii) *if f is a face of G and $bd(f) \subseteq G'$, then f is a face of G', and*

(iii) *if $f' \in F(G')$ is disjoint from G, then $f' \in F(G)$.*

Proof. Statement (i) follows immediately: a face f of G is an arcwise-connected subset of $\mathbb{R}^2 - G$, and hence is a subset of an arcwise-connected component of $\mathbb{R}^2 - G'$, that is, a face of G'.

To prove (ii) let f be a face of G. By (i) there exists a face f' of G' such that $f \subseteq f'$. If $f = f'$, then (ii) holds, and so we may assume that the inclusion is proper. Hence there exists a point $x' \in f' - f$. Let $x \in f$, and let $A \subseteq f'$ be a polygonal arc with ends x, x'. Since A is not a subset of f, it intersects the boundary of f, say in a point z. But $z \in f'$, and hence $z \notin G'$, contrary to hypothesis. This proves (ii).

To prove (iii) let $f' \in F(G')$ be disjoint from G. Then f' is an arcwise connected subset of $\mathbb{R}^2 - G$, and therefore is a subset of a face f of G. By (i) we have $f \subseteq f'$, and hence $f' = f \in F(G)$, as desired. □

Lemma 1.11. *Let* e *be an edge of a plane graph* G *that belongs to a cycle of* G*, and let* $f_1, f_2 \in F(G)$ *be the two faces incident with* e*. Then* $f_1 \neq f_2$*.*

Proof. For $i = 1, 2$ the face f_i is a subset of a face f'_i of C, by Lemma 1.10(i). By Lemma 1.6 the edge e is disjoint from the boundary of all faces of $F(G) - \{f_1, f_2\}$, and hence it is disjoint from the boundary of all faces of $F(C) - \{f'_1, f'_2\}$. But the edge e belongs to the boundary of both faces of C by Theorem 1.3, and hence $f'_1 \neq f'_2$, implying that $f_1 \neq f_2$, as desired. □

Lemma 1.12. *Let* G *be a plane graph, let* $e \in E(G)$*, and let* f_1 *and* f_2 *be the two faces incident with* e*. Let* f_{12} *denote the point set* $f_1 \cup e \cup f_2$*. Then* f_{12} *is a face of* $G\backslash e$ *and* $F(G) - \{f_1, f_2\} = F(G\backslash e) - \{f_{12}\}$*.*

Proof. To prove the first assertion we notice that f_{12} is arcwise connected, and hence is a subset of a face f' of $G\backslash e$. We may assume that f_{12} is a proper subset of f', for otherwise $f_{12} \in F(G\backslash e)$, as desired. Thus there exists a polygonal arc A with ends $x \in f_{12}$ and $y \in f' - f_{12}$ such that $A \subseteq f'$. By considering a proper subset of A we may assume that $A \cap e = \emptyset$. Since $y \in f' - f_{12} \subseteq \mathbb{R}^2 - G - f_1 - f_2$ it follows that y belongs to a face of G other than f_1 or f_2, but then A connects points belonging to different faces of G, and yet $A \cap G = \emptyset$, a contradiction. Thus $f_{12} \in F(G\backslash e)$.

For the second assertion let $f \in F(G) - \{f_1, f_2\}$. Since f_1, f_2 are the only two faces of G incident with e by Lemma 1.6, we deduce that $\mathrm{bd}(F) \subseteq G\backslash e$, and hence $f \in F(G\backslash e)$ by Lemma 1.10(ii). Clearly $f \neq f_{12}$. Conversely, let $f' \in F(G\backslash e) - \{f_{12}\}$. Then $f' \cap G = \emptyset$, and hence $f' \in F(G)$ by Lemma 1.10(iii). Further, $f' \notin \{f_1, f_2\}$, because $f_1, f_2 \notin F(G\backslash e)$. □

Theorem 1.13 (Euler's formula). *Every connected plane simple graph* G *satisfies* $|V(G)| + |F(G)| = |E(G)| + 2$*.*

Proof. We proceed by induction on $|E(G)|$. If G has no cycles, then it is a tree, and the formula holds by Lemma 1.9. Thus we may assume that G has a cycle C. Let $e \in E(C)$, and let $f_1, f_2 \in F(G)$ be the two faces incident with e. Then $f_1 \neq f_2$ by Lemma 1.11. The formula follows from Lemma 1.12 by induction applied to the graph $G\backslash e$. □

Corollary 1.14. *Every simple planar graph* G *on* $n \geq 3$ *vertices has at most* $3n-6$ *edges. Moreover, if* G *has no triangles, then it has at most* $2n - 4$ *edges.*

Proof. We may assume that G is connected, for an edge joining vertices in different components of G may be added without violating planarity. Let q be the number edge-face incidences (e, f), with the proviso that if f_1 and f_2 are the two faces incident with e and $f_1 = f_2$, then the incidence $(e, f_1) = (e, f_2)$ is counted twice. Then $q = 2|E(G)|$. On the other hand, since G has no loops or parallel edges, each face contributes at least three toward q, and hence $q \geq 3|F(G)|$. Thus $|F(G)| \leq 2|E(G)|/3$, and substituting this into Euler's formula gives the the first inequality. The second inequality follows similarly. □

Corollary 1.15. *The graphs* K_5 *and* $K_{3,3}$ *are not planar.*

Proof. This follows immediately from Corollary 1.14. □