WEEK 1 PROBLEMS
Math 6014A

1. Let $n \geq 1$ be an integer. Prove that a sequence of positive integers $d_1, d_2, \ldots, d_n$ is a degree sequence of a tree if and only if $d_1 + d_2 + \cdots + d_n = 2n - 2$.

2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of points in the plane such that the distance between any two distinct points in $S$ is at least one. Show that there are at most $3n$ pairs of points at distance exactly one.

3. If a graph has minimum degree $\delta \geq 2$, then it contains a cycle of length at least $\delta + 1$.
   \textit{Hint.} Consider a longest path.

4. The \textit{girth} of a graph $G$ is the length of a shortest cycle in $G$; if $G$ has no cycles we define the girth of $G$ to be infinite. Show that for $k \geq 2$
   (a) a $k$-regular graph of girth 4 has $\geq 2k$ vertices, and up to isomorphism there is exactly one such graph on $2k$ vertices,
   (b) a $k$-regular graph of girth 5 has $\geq k^2 + 1$ vertices,
   (c) a $k$-regular graph of girth 5 in which every two vertices have distance $\leq 2$ has exactly $k^2 + 1$ vertices, and find such a graph for $k = 2, 3$. (It is known that such a graph can only exist for $k = 2, 3, 7$, and, possibly, 57.)
   \textit{Hint.} Take a vertex $v \in V(G)$, look at all vertices adjacent to $v$, and look at all vertices adjacent to these vertices.

5. Let $G$ be a connected graph, and let $r \in V(G)$. Prove that $G$ has a spanning tree $T$ such that for every edge of $G$ with ends $u$ and $v$, either $u$ belongs to the unique path in $T$ with ends $v$ and $r$, or $v$ belongs to the unique path in $T$ with ends $u$ and $r$. (This is called a \textit{depth first search tree} with root $r$.)

6. Show that if a graph $G$ contains $k$ edge-disjoint spanning trees, then for each partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$, the number of edges of $G$ which have ends in different parts of the partition is at least $k(n-1)$. (Tutte and Nash-Williams have shown that the converse is also true.)