WEEK 13 PROBLEMS

Math 6014A

1. A *tournament* is a directed graph T such that for every two distinct vertices $u, v \in V(T)$, T contains exactly one of the directed edges $(u, v), (v, u)$. (This corresponds to the results of a tournament: $(u, v) \in E(T)$ means u beat v, whereas $(v, u) \in E(T)$ means v beat u.) Prove that there is a tournament T on n vertices with at least $n!2^{1-n}$ (directed) Hamiltonian paths.
Hint. Consider a random tournament on n vertices. Thus for every unordered pair $u, v \in V(T)$ of distinct vertices, $(u, v) \in E(T)$ with probability $1/2$ and $(v, u) \in E(T)$ with probability $1/2$, and the events $(u, v) \in E(T)$ are mutually independent. Compute the expected number of Hamiltonian paths.

2. Let G be a simple graph with n vertices and m edges. Prove that G contains a bipartite subgraph with at least $m/2$ edges.
Hint. Let A be a random subset of $V(G)$, that is $P[v \in A] = 1/2$ and these probabilities are mutually independent. Let $B = V(G) - A$. Compute the expected number of edges with one end in A and the other end in B.

3. Let $0 < p < 1$ and $\epsilon > 0$ be fixed. Prove that almost every graph in $\mathcal{G}(n, p)$ has at least $(p - \epsilon)n^2/2$ edges and at most $(p + \epsilon)n^2/2$ edges.

4. Let $p = c(\log n)/n$. Prove that if $c > 1$ then a.e. graph has no isolated vertices, and that if $c < 1$ then a.e. graph has an isolated vertex.

5. Let $p = c(\log n)/n$. Prove that if $c > 1$ then a.e. graph is connected, and that if $c < 1$ then almost no graph is connected.
Hint. Use 4 and compute the probability that G has a component of at least two and at most $n/2$ vertices.

6. Let $f(d) = \binom{n}{d}2^{-\binom{d}{2}}$. Prove that for $d \geq \log_2 n$ this function is decreasing, and deduce that there exists an integer $d_0 = d_0(n)$ such that $f(d_0) \geq \log n > f(d_0 + 1)$. Prove that there exists a constant c such that $2\log_2 n - c \log\log n \leq d_0 \leq 2\log_2 n + 1$.