WEEK 15 PROBLEMS

Math 6014

1. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Prove that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.

2. Let G be a simple 3-connected non-planar graph. Prove that either G is isomorphic to K_5 or G has a subgraph isomorphic to a subdivision of $K_{3,3}$.
Hint. By Kuratowski's theorem we may assume that G has a subgraph H isomorphic to a subdivision of K_5. That is, H consists of five vertices $v_1, v_2, \ldots, v_5$ and ten paths P_{ij}, where $i, j = 1, 2, \ldots, 5$ and $i < j$ in such a way that P_{ij} has endpoints v_i, v_j and different paths are vertex-disjoint except possibly at their endpoints. If one of the paths, say P_{12}, has more than two vertices, then use the fact that $\{v_1, v_2\}$ is not a cutset of G. Otherwise H is isomorphic to K_5.

3. A graph is *outerplanar* if it is isomorphic to a plane graph such that every vertex is incident with the unbounded face. Prove that a graph is outerplanar if and only if it has no subgraph isomorphic to a subdivision of K_4 or $K_{2,3}$.

4. Let G be a multigraph. Prove that the following statements are equivalent:
(i) G is a block
(ii) every two edges lie on a common cycle
(iii) every two edges belong to common minimal cut

5. Let G, G^* be a pair of dual plane multigraphs. Prove that G is a block if and only if G^* is a block.