WEEK 2 PROBLEMS

Math 6014A

1. Let $k \geq 1$ be an integer, and let T be a tree on $k+1$ vertices. Show that if a graph G has minimum degree at least k, then G has a subgraph isomorphic to T.

2. Let t_1, t_2, t_3 be vertices of a tree T. Prove that there is a unique vertex t of T such that for every $i, j = 1, 2, 3$ with $i \neq j$ the vertex t lies on the unique path between t_i and t_j in T.

3. Let $T_1, T_2, \ldots, T_k$ be subtrees of a tree such that any two of them have a vertex in common. Prove that they all have a vertex in common.

4. Let e be an edge of K_n. Show that $\tau(K_n \backslash e) = (n-2)n^{n-3}$.

5. Let D be an orientation of a graph G, let $V(D) = \{v_1, v_2, \ldots, v_n\}$, $E(D) = \{e_1, e_2, \ldots, e_m\}$, and let $M = (m_{i,j})$ be the incidence matrix of D. That is, $m_{i,j} = 1$ if the vertex v_i is the tail of the edge e_j, $m_{i,j} = -1$ if the vertex v_i is the head of the edge e_j, and $m_{i,j} = 0$ otherwise.

(i) Prove that MM^T is the Laplace matrix of G.

(ii) Let K be obtained from M by deleting row k. Prove that for $S \subseteq \{1, 2, \ldots, m\}$ we have $|\det(K|S)| = 1$ if $\{e_j : j \in S\}$ is the edge-set of a spanning tree in G, and $\det(K|S) = 0$ otherwise. ($K|S$ denotes the matrix consisting of columns of K that are indexed by an element of S.)

(iii) Deduce the matrix-tree theorem, using the theorem below.

The Binet-Cauchy Theorem. *Let A be an $n \times m$ matrix, where $n \leq m$. Then*

$$\det(AA^T) = \sum \left(\det(A|S)\right)^2,$$

the summation taken over all subsets S of the columns of A of size n.