WEEK 3 PROBLEMS

Math 6014A

1. Let T_1 and T_2 be two spanning trees of a connected graph G. Prove that T_1 can be transformed to T_2 through a sequence of intermediate trees, each arising from the previous one by deleting an edge and adding another one.

2. Let T_1 and T_2 be two spanning trees of a 2-connected graph G. Prove that T_1 can be transformed to T_2 through a sequence of intermediate trees, each arising from the previous tree, say T, by deleting an edge incident with a vertex t of degree one in T, and adding another edge incident with t.

3. A *circulation* in a directed graph D is a function $g : E(D) \to \mathbf{R}$ satisfying the conservation condition at every vertex. Let $l, u : E(D) \to \mathbf{R}_0^+$ be a *lower capacity function* and *upper capacity function*, respectively, and assume that $l(e) \leq u(e)$ for every edge $e \in E(D)$. A circulation g is *feasible* if $l(e) \leq g(e) \leq u(e)$ for every edge $e \in E(D)$. Prove that there exists a feasible circulation if and only if $l^+(A) \leq u^-(A)$ for every set $A \subseteq V(D)$.
Hint. Add a vertex s and an edge from s to every vertex of D, and a vertex t and an edge from every vertex of D to t. Define $c(sv) = l^-(v)$, $c(vt) = l^+(v)$, and $c(e) = u(e) - l(e)$ for $v \in V(D)$ and $e \in E(D)$. Given a flow f of value $\sum_{e \in E(D)} l(e)$ in the network thus defined, consider $f + l$.

4. Let f be a flow in a network $N = (D, c, s, t)$, and let f' be obtained from a shortest f-augmenting path as in the proof of the Max-Flow Min-Cut theorem. Define a digraph D_f by saying that $V(D_f) = V(D)$ and $\overrightarrow{uv} \in E(D_f)$ if either $\overrightarrow{uv} \in E(D)$ and $f(\overrightarrow{uv}) < c(\overrightarrow{uv})$, or $\overrightarrow{vu} \in E(D)$ and $f(\overrightarrow{vu}) > 0$. Prove that for every $v \in V(D)$ we have $\mathrm{dist}_{D_{f'}}(s, v) \geq \mathrm{dist}_{D_f}(s, v)$ and $\mathrm{dist}_{D_{f'}}(v, t) \geq \mathrm{dist}_{D_f}(v, t)$.

5. Let N, f and f' be as in the previous problem, and assume that the shortest f-augmenting path has length k. Let E_f denote the set of all edges that belong to an f-augmenting path of length k. Prove that $E_{f'}$ is a proper subset of E_f.

6. Deduce that by choosing an augmenting path of minimum length at every step, the process from the proof of the Max-Flow Min-Cut theorem will result in a maximum flow after at most $|V(D)| \cdot |E(D)|$ augmentations.