WEEK 7 PROBLEMS

Math 6014A

1. Let G be a graph with n vertices and at most $r \geq 2$ independent vertices (= no two are adjacent). Prove that if D is an orientation of G which does not contain a directed cycle then D contains a directed path of length at least $\lceil \frac{n}{r} \rceil - 1$.

2. An $r \times s$ *Latin rectangle* based on $1, 2, \ldots, n$ is an $r \times s$ matrix $A = (a_{ij})$ such that each entry is one of the integers $1, 2, \ldots, n$ and each integer occurs in each row and column at most once. Prove that every $r \times n$ Latin rectangle A can be extended to an $n \times n$ *Latin square.* [*Hint.* Assume that $r < n$ and extend A to an $(r+1) \times n$ Latin rectangle. Let A_j be the set of possible values $a_{r+1,j}$, that is let $A_j = \{k : 1 \leq k \leq n, k \neq a_{ij}\}$. Check that $\{A_j : 1 \leq j \leq n\}$ has a set of distinct representatives.] Prove that there are at least $n!(n-1)! \cdots (n-r+1)!$ distinct $r+n$ Latin rectangles based on $1, 2, \ldots, n$.

3. Let A be an $r \times s$ Latin rectangle based on $1, 2, \ldots, n$ and denote by $A(i)$ the number of times the symbol i occurs in A. Show that A can be extended to an $n \times n$ Latin square if and only if $A(i) \geq r + s - n$ for every $i = 1, 2, \ldots, n$.

4. (Dual of Dilworth's Theorem) Let $(P, \leq)$ be a partially ordered set such that every chain has at most k elements. Prove that P can be decomposed into k antichains.

5. Let G be a graph, let M be a matching in G, and let C be a cycle in G of length $2k+1$ such that $E(C)$ includes exactly k edges of M. Assume also that exactly one vertex of C is not incident with any edge of M. Let G' be obtained from G by contracting all edges of C and deleting all resulting loops and parallel edges, and let $M' = M - E(C)$. Prove that M is a maximum matching in G if and only if M' is a maximum matching in G'.

6. Let G be a graph, let M be a matching in G, and let $v \in V(G)$ be incident with no edge of M. Let us say that a vertex $u \in V(G)$ is *even* if there exists an M-alternating uv-path of even length, and let us define *odd* vertices analogously. Prove that if a vertex is both odd and even, then there exists a cycle and a matching as in the previous problem. Explain how to determine whether there is an M-augmenting path with one end v, assuming no vertex is both odd an even.

The previous two problems give rise to a polynomial-time algorithm to find a maximum matching in a graph.