WEEK 8 PROBLEMS
Math 6014A

1. Prove that the edge-chromatic number of a bipartite multigraph G is $\Delta(G)$.

2. A set $F \subseteq E(G)$ is called an *edge cover* if every vertex of G is incident with at least one edge in F. Let G be a bipartite graph with minimum degree at least one. Prove that the size of a maximum independent set is equal to the size of a minimum edge cover.

3. Prove that bipartite graphs and their complements are perfect.

4. Prove that line graphs of bipartite graphs and their complements are perfect. Do not use the perfect graph theorem.

5. A graph G is a *comparability graph* if there exists a poset $(P, \leq)$ such that $V(G) = P$ and two distinct vertices $u, v \in V(G)$ are adjacent if $u \leq v$ or $v \leq u$. Prove that comparability graphs and their complements are perfect.

6. For an integer $k \geq 0$ let $p_G(k)$ be the number of proper colorings of a multigraph G using k colors. Prove that for every non-loop edge e of G and every integer $k \geq 0$,

$$p_G(k) = p_{G \backslash e}(k) - p_{G/e}(k).$$

Deduce that $p_G(k)$ is a polynomial in k. It is called the *chromatic polynomial.*

7. Let G be a connected simple graph. Prove that $p_G(k) \leq k(k-1)^{|V(G)|-1}$ for all integers $k \geq 0$. Prove that equality holds for all integers $k \geq 0$ if and only if G is a tree.

8. For every positive integer k construct a simple graph G with $\chi(G) \geq k$ and no triangles.
Hint. If G_k works for k, construct G_{k+1} as follows. Let $V(G_k) = \{v_1, v_2, \ldots, v_n\}$, add $n+1$ new vertices $\{u, u_1, u_2, \ldots, u_n\}$, and for $i = 1, 2, \ldots, n$ join u_i to the neighbors of v_i and to u.