CHAPTER 3

Some equivalent problems

3.1. Pólya's permanent problem

Computing the permanent of a matrix seems to be of a different computational complexity from computing the determinant. While the determinant can be calculated using Gaussian elimination, no efficient algorithm for computing the permanent is known, and, in fact, none is believed to exist. More precisely, Valiant [**84**] has shown that computing the permanent is #P-complete even when restricted to 0-1 matrices.

It is therefore reasonable to ask if perhaps computing the permanent can be somehow reduced to computing the determinant of a related matrix. In particular, Pólya [**61**] proposed in 1913 to show that the related matrix of every 0-1 square matrix cannot be obtained by changing some of the 1's to -1's. Pólya's question was settled by Szegö [**73**]; the proof is not hard and we leave it as the next exercise. What is harder, though, and is a subject of our study, is for which matrices does such a related matrix exist. Thus we make the following definition. Given a non-negative square matrix A, let us say that a matrix B is a *Pólya matrix* for A if B is obtained from A by changing the signs of some of the entries of A and $\det(A) = \mathrm{perm}(B)$. Here is a simple example showing that not every 0-1 square matrix has a Pólya matrix.

Exercise 3.1.1. Prove that the 3×3 matrix of all ones has no Pólya matrix.

Now we can restate Pólya's problem as a combinatorial decision problem as follows.

3.1.2. PÓLYA'S PERMANENT PROBLEM
Instance: A square 0-1 matrix A
Question: Is there a Pólya matrix for A?

Vazirani and Yannakakis [**85**] proved that PÓLYA'S PERMANENT PROBLEM is polynomial-time equivalent to BIPARTITE PFAFFIAN ORIENTATION. In fact, they proved the a more specific result, but we need a couple of definitions and an exercise before we can prove it. Let G be a bipartite graph with bipartition (X, Y). The *bipartite adjacency matrix* of G has rows indexed by X, columns indexed by Y, and the entry in row x and column y is 1 or 0 depending on whether x is adjacent to y or not. If G is directed, then the *directed bipartite adjacency matrix* of G has rows indexed by X, columns indexed by Y, and the entry in row x and column y is 1, -1 or 0 depending on whether G has an edge directed from x to y, or G has an edge directed from y to x, or x and y are not adjacent in G.

Exercise 3.1.3. Let $n \geq 1$ be an integer, let G be a simple bipartite graph with bipartition $\{1, 3, \ldots, 2n-1\}, \{2, 4, \ldots, 2n\}$, let D be an orientation of G, and let

$B = (b_{ij})$ be the $n \times n$ matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } (2i-1, 2j) \in E(D) \\ -1 & \text{if } (2j, 2i-1) \in E(D) \\ 0 & \text{otherwise} \end{cases}$$

Let M be a perfect matching in G, and let σ be the permutation of $[n]$ defined by saying that M consists of all edges of the form $\{2i-1, 2\sigma(i)\}$. Then

$$\mathrm{sgn}_D(M) = \mathrm{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdot \ldots \cdot b_{n\sigma(n)}$$

Now we are ready to state and proof the result of Vazirani and Yannakakis [**85**].

Theorem 3.1.4. *Let G be a bipartite graph with bipartite adjacency matrix A, let D be an orientation of G, and let B be the directed bipartite adjacency matrix of D. Then* $\mathrm{per}(A) = |\det(B)|$ *if and only if D is a Pfaffian orientation of G.*

Proof. By Exercise 2.2.5 the orientation D is Pfaffian if and only if $\mathrm{sgn}_D(M) = \mathrm{sgn}_D(M')$ for every two perfect matchings M, M' of G. The latter statement is equivalent to $\mathrm{per}(A) = |\det(B)|$ by Exercise 3.1.3. □

The following two corollaries follow immediately. Exercise 3.1.7 shows that the same conclusions hold for non-negative matrices.

Corollary 3.1.5. *A bipartite graph is Pfaffian if and only if its bipartite adjacency matrix has a Pólya matrix.*

Corollary 3.1.6. *Let G be a bipartite graph with bipartite adjacency matrix A and bipartition (X, Y), and let D be the orientation of G obtained by orienting all edges of G from X to Y. Then D is a Pfaffian orientation of G if and only if* $\mathrm{perm(A)} = |\det(A)|$.

Exercise 3.1.7. Let A be an $n \times n$ matrix with nonnegative entries, and let G be the bipartite graph with vertices corresponding to rows and columns of A such that a vertex corresponding to row r is adjacent to a vertex corresponding to column c if and only if $a_{rc} \neq 0$. Then A has a Pólya matrix if and only if G has a Pfaffian orientation.

3.2. Even digraphs

Let us turn to directed graphs now. We are interested in the following problem.

3.2.1. EVEN DIRECTED CYCLE
Instance: A directed graph D
Question: Does D have an even cycle?

It is perhaps surprising that EVEN DIRECTED CYCLE is not trivial, unlike its undirected counterpart or finding an odd directed cycle. Some indication of the potential difficulty can be gleaned from the fact that the digraph F_7 pictured in Figure 3.2 has no even cycle.

Exercise 3.2.2. Prove that the digraph F_7 has no even cycle.

It turns out that it is more profitable to study the following polynomial-time equivalent problem. A digraph D is *even* if for every weight function $w : E(D) \to \{0, 1\}$ there exists a cycle in D of even total weight.

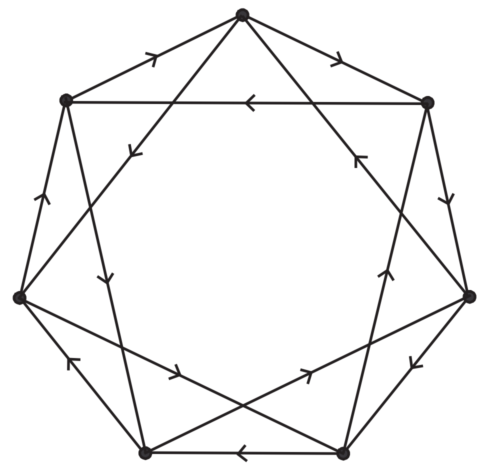

FIGURE 1. The digraph F_7

3.2.3. EVEN DIGRAPH
Instance: A directed graph D
Question: Is D even?

We will show that the above two problems are polynomial-time equivalent. For the proof we need a couple of exercises.

Exercise 3.2.4. Every strongly connected digraph has a cycle basis $\mathcal{B}$ consisting of directed cycles, and there is a weight function $w : E(D) \to \{0, 1\}$ such that the total weight of every cycle in $\mathcal{B}$ is odd.

The above exercise suffices for our purposes, but a stronger claim can be made.

Exercise 3.2.5. For every strongly connected digraph and every cycle basis $\mathcal{B}$ consisting of directed cycles there is a weight function $w : E(D) \to \{0, 1\}$ such that the total weight of every cycle in $\mathcal{B}$ is odd.

Exercise 3.2.6. Let D be a strongly connected digraph, let $w : E(D) \to \{0, 1\}$ be a weight function, and let D have a cycle basis $\mathcal{B}$ such that every member of B is a directed cycle of odd total weight. Then D is even if and only if D has a cycle of even total weight.

The polynomial-time equivalence of Problems 3.2.1 and 3.2.3 was shown by Seymour and Thomassen [**72**]. We will find out later in this section that it is a special case of Theorem 2.5.16.

Theorem 3.2.7. EVEN DIRECTED CYCLE *is polynomial-time equivalent to* EVEN DIGRAPH.

Proof. First, let D be an instance of EVEN DIGRAPH. Since we may assume that D is strongly connected, by Exercise 3.2.4 we find a cycle basis $\mathcal{B}$ consisting of directed cycles and a weight function $w : E(D) \to \{0, 1\}$ such that the total weight of every cycle in $\mathcal{B}$ is odd. Let D' be obtained from D by subdividing each edge of zero weight exactly once. By Exercise 3.2.6 the digraph D is even if and only if D' has a cycle of even total weight.

Conversely, let D be an instance of EVEN DIRECTED CYCLE. Again, we may assume that D is strongly connected, and we may select a cycle basis $\mathcal{B}$ consisting of directed cycles. We may assume that every cycle in $\mathcal{B}$ is odd, for otherwise we are done. By applying Exercise 3.2.6 to the weight function that assigns 1 to every edge we deduce that D is even if and only if it has an even cycle, as desired. $\square$

Next we show that EVEN DIGRAPH is polynomial-time equivalent to BIPARTITE PFAFFIAN ORIENTATION. Let us recall that for a bipartite graph G with bipartition (A, B), and a perfect matching M of G we denote by $D(G, M)$ the digraph obtained from G by directing every edge from A to B, and contracting every edge of M. We need the following exercise.

Exercise 3.2.8. Let G be a bipartite graph with bipartition (A, B) and perfect matching M, let D be an orientation of G such that every edge of M is directed from A to B, let $D' := D(G, M)$, let C be an M-alternating cycle in G, let C' be the corresponding directed cycle of D', and let $w : E(D') \to \{0, 1\}$ be defined by $w(e) = 1$ if e (regarded as an edge of G) is directed in D from A to B and $w(e) = 0$ otherwise. Then C is oddly oriented in D if and only if $w(C')$ is odd.

We are now ready to prove a result of Little [**39**] that EVEN DIGRAPH is polynomial-time equivalent to BIPARTITE PFAFFIAN ORIENTATION. More precisely, Little proved the following.

Theorem 3.2.9. *Let G be a bipartite graph, and let M be a perfect matching in G. Then G has a Pfaffian orientation if and only if $D(G, M)$ is not even.*

Proof. For an orientation D of G let w be as in Exercise 3.2.8 and vice versa. It follows from that exercise that $D(G, M)$ has no cycle of even weight if and only if every M-alternating cycle of G is oddly oriented in D. The latter is equivalent to D being a Pfaffian orientation of G by Exercise 2.2.6. $\square$

Similarly, we deduce the following.

Exercise 3.2.10. Let G be a bipartite graph with bipartition (A, B), let M be a perfect matching of G, and let D be the orientation of G obtained by directing all edges from A to B. Then D is a Pfaffian orientation of G if and only if $D(G, M)$ has no even cycle.