In light of Theorem 3.2.9 one expects a characterization of even digraphs, analogous to Theorem 2.6.2. It is perhaps somewhat surprising that such a theorem involves infinitely many obstructions, rather than one. To state the theorem we need a couple of definitions and exercises.

Let D be a digraph. By an *odd double cycle* we mean the directed graph obtained from an odd cycle by replacing each edge by two directed edges, one in each direction. By *splitting a vertex of a digraph D* we mean replacing a vertex v by two vertices v_1 and v_2 such that v_1v_2 is a directed edge of the new graph, every edge that used to have head v now has head v_1 and every edge that used to have tail v now has tail v_2. Finally, by a *weak odd double cycle* we mean any digraph that can be obtained from an odd double cycle by repeatedly splitting vertices.

Exercise 3.2.11. Prove that every weak odd double cycle is even.

We are now ready to state the characterization of even digraphs due to Seymour and Thomassen [**72**]. A shorter proof using Theorem 2.6.2 is given by McCuaig [**48**]. Our proof takes advantage of the argument of Norine, Little and Teo [**56**] used to prove Theorem 2.6.2.

Theorem 3.2.12. *A digraph is even if and only if it has a subdigraph isomorphic to a subdivision of an odd double cycle.*

Proof. The "if" part follows from Exercise 3.2.11. To prove "only if" let D be an even digraph. Then there is a bipartite graph G and a perfect matching M in G such that $D = D(G, M)$. By Theorem 3.2.9 the graph G has no Pfaffian orientation, and hence by Exercise 2.6.3 it has a subgraph isomorphic to a subdivision of a Möbius ladder with an odd number of rungs such that H is M-covered, and every rung of H is M-covered. It follows that $D(H, M)$ is a weak odd double cycle in D, as required. □

Exercise 3.2.13. Let G be a bipartite graph with bipartition (A, B) and let D be the orientation of G obtained by directing every edge from A to B. Then D is a Pfaffian orientation of G if and only if G has no central cycle of length divisible by four.

Exercise 3.2.14. Let D be an orientation of a bipartite graph with bipartition (A, B), let F be the set of all edges of D that are directed from B to A, and let C be a cycle in D. Then C is oddly oriented if and only if $|E(C) \cap F| + |E(C)|/2$ is odd.

Exercise 3.2.15. Prove that the following problem is polynomial-time equivalent to EVEN DIRECTED CYCLE:
Instance: A directed graph D
Question: Does there exist a cycle in D whose edge-set can be written as a symmetric difference of the edge-sets of an even number of cycles of D?

Exercise 3.2.16. Prove that it is NP-hard to decide whether a digraph has an even cycle containing a given edge.

3.3. An economics example

To motivate the topic of the next section we present a simple example from Brualdi and Shader [**7**] of a single-commodity trade. The example is similar to one discussed by Samuelson [**68**].

Following [**7**] we consider a market for bananas, where the following variables come into play: the supply S of bananas, the demand D for bananas, the price p of bananas, and a parameter t, interpreted as people's taste in bananas. We assume that the supply $S = S(p)$ depends on price only, and that if price increases, then so will increase the supply, because farmers will produce more bananas. Further, we assume the demand $D = D(p, t)$ depends on price and people's taste, that demand increases as people's taste in bananas increases, and that demand increases as price decreases. Thus we have

$$\frac{dS}{dp} > 0, \quad \frac{\partial D}{\partial t} > 0, \quad \frac{\partial D}{\partial p} < 0. \tag{3.1}$$

The price $p = p(t)$ depends on people's taste. Let $x = x(t)$ denote the amount of bananas on the market, given that taste is t. We assume market equilibrium; that is

$$S(p) - x(t) = 0 \tag{3.2}$$

$$D(p, t) - x(t) = 0. \tag{3.3}$$

By differentiating with respect to t we obtain, in matrix terms

$$\begin{bmatrix} \frac{dS}{dp} & -1 \\ \frac{\partial D}{\partial p} & -1 \end{bmatrix} \begin{bmatrix} \frac{dp}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial D}{\partial t} \end{bmatrix} \tag{3.4}$$

Inequalities (3.1) imply that the system (3.4) has a unique solution satisfying

$$\frac{dp}{dt} > 0 \text{ and } \frac{dx}{dt} > 0. \tag{3.5}$$

Thus we have proven that the economic assumptions made earlier imply that both the price and the amount of bananas on the market increase as people's taste for bananas increases. The next section discusses the matrix property that made the conclusion possible.

3.4. Sign-nonsingular matrices

We say that two real $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ have the same *sign pattern* if for all pairs of indices i, j the entries a_{ij} and b_{ij} have the same sign; that is, they are both strictly positive, or they are both strictly negative, or they are both zero. A real $m \times m$ matrix A is *sign-nonsingular* if every real $m \times m$ matrix with the same sign pattern is non-singular. Thus we have the following decision problem.

3.4.1. SIGN-NONSINGULARITY
Instance: A square matrix A
Question: Is A sign-nonsingular?

The next exercise shows that SIGN-NONSINGULARITY is a combinatorial problem.

Exercise 3.4.2. A matrix A is sign-nonsingular if and only if some term in the determinantal expansion of A is non-zero, and every two such terms have the same sign.

There is a related notion for rectangular matrices. We say that a real $m \times n$ matrix is an *L-matrix* if every $m \times m$ submatrix of A is sign-nonsingular. Klee, Ladner and Manber [**35**] proved that it is NP-hard to test whether an $m \times n$ matrix is an L-matrix, but the proof does not apply to square matrices.

Maybee [**46**] proved that SIGN-NONSINGULARITY is polynomial-time equivalent to EVEN DIRECTED CYCLE. We show polynomial-time equivalence to BIPARTITE PFAFFIAN ORIENTATION as follows.

Theorem 3.4.3. *Let G be a bipartite graph, let D be an orientation of G, and let A be the directed bipartite adjacency matrix of D. Then A is sign-nonsingular if and only if G has a perfect matching and D is a Pfaffian orientation of G.*

Proof. By Exercise 3.4.2 the matrix A is sign-nonsingular if and only if some term in the determinantal expansion of A is non-zero, and every two such terms have the same sign. Some term in the determinantal expansion of A is non-zero if and only if G has a perfect matching, and by Exercises 3.1.3 every two non-zero terms have the same sign if and only if D is a Pfaffian orientation of G. □

As the example in the previous section indicates, in economic analysis one may not know the exact quantitative relationships between different variables, but there may be some qualitative information such as that one quantity rises if and only if another does. Thus we may want to deduce qualitative information about the solution to a linear system $A\mathbf{x} = \mathbf{b}$ from the knowledge of the sign-patterns of the matrix A and vector $\mathbf{b}$. That motivates the following definition. We say that the linear system $A\mathbf{x} = \mathbf{b}$ is *sign-solvable* if for every real matrix B with the same sign-pattern as A and every vector $\mathbf{c}$ with the same sign-pattern as $\mathbf{b}$ the system $B\mathbf{y} = \mathbf{c}$ has a unique solution $\mathbf{y}$, and its sign-pattern does not depend on the choice of B and $\mathbf{c}$. The study of sign-solvability was first proposed by Samuelson [**68**]. We formalize it as follows.

3.4.4. SQUARE SIGN-SOLVABILITY
Instance: An $n \times n$ matrix A and a vector $\mathbf{b} \in \mathbf{R}^n$
Question: Is the linear system $A\mathbf{x} = \mathbf{b}$ sign-solvable?

It follows from standard linear algebra that square sign-solvability can be decided efficiently if and only if sign-nonsingularity can. We leave that as an exercise.

Exercise 3.4.5. Prove that SIGN-NONSINGULARITY is polynomial-time equivalent to SQUARE SIGN-SOLVABILITY.

3.5. The polytope of even permutation matrices

A square 0-1 matrix A is a *permutation matrix* if every row and every column of A contain exactly one 1. A real matrix with non-negative entries is *doubly stochastic* if all row sums and all column sums are 1. A classical theorem of Birkhoff [**4**] characterizes the convex hull of permutation matrices; we state it as an exercise.

Exercise 3.5.1. A matrix belongs to the convex hull of permutation matrices if and only if it is doubly stochastic.

If $A = (a_{ij})$ is an $n \times n$ permutation matrix, then there is a unique permutation σ of $[n]$ such $a_{ij} = 1$ if and only if $j = \sigma(i)$. We say that A is an *even permutation matrix* if σ is an even permutation. Let $\mathcal{Q}(n)$ denote the convex hull of even permutation matrices. This polytope has been studied, but the complexity status of the following decision problem is not known.

3.5.2. THE EVEN PERMUTATION MATRIX POLYTOPE MEMBERSHIP
Instance: A rational $n \times n$ matrix A
Question: Does $A \in \mathcal{Q}(n)$?

Cunningham and Wang [**11**] pointed out that by a fundamental result of Grötschel, Lovász and Schrijver [**25**], the membership problem 3.5.2 is solvable in polynomial time if there is a polynomial-time algorithm for the following optimization problem, where $C \cdot X$ denotes $\sum_{i,j=1}^{n} c_{ij}x_{ij}$.

3.5.3. OPTIMIZATION OVER THE EVEN PERMUTATION MATRIX POLYTOPE
Instance: A rational $n \times n$ matrix C
Objective: Find the maximum of $C \cdot X$ over all $X \in \mathcal{Q}(n)$.

We are interested in a special case of Problem 3.5.3 when C is a 0-1 matrix and we want to determine whether the maximum is n. In that case let G be a bipartite graph with bipartite adjacency matrix C. Since a maximum of Problem 3.5.3 is attained at a vertex of $\mathcal{Q}(n)$, we deduce that the maximum is n if and only if G has an *even perfect matching*, by which we mean a perfect matching such that the corresponding permutation (say as in Exercise 3.1.3) is even. Thus the following is polynomial-time equivalent to the special case of Problem 3.5.3 when C is a 0-1 matrix and we want to determine whether the maximum is n.

3.5.4. EVEN PERFECT MATCHING
Instance A bipartite graph G
Question Does G have no even perfect matching?

We now show that Problem 3.5.4 is polynomial-time equivalent to BIPARTITE PFAFFIAN ORIENTATION. For that we need an exercise.

Exercise 3.5.5. Let G be a bipartite graph with bipartition (X, Y), and let D be the orientation of G obtained by orienting each edge from X to Y. Then D is Pfaffian if and only if either every perfect matching of G is even, or every perfect matching of G is not even.

Theorem 3.5.6. *The problems EVEN PERFECT MATCHING and BIPARTITE PFAFFIAN ORIENTATION are polynomial-time equivalent.*

Proof. By Corollary 2.5.17 it suffices to prove the polynomial time equivalence of EVEN PERFECT MATCHING and the restriction of IS ORIENTATION PFAFFIAN to bipartite graphs. Let G be an instance of EVEN PERFECT MATCHING, let (X, Y) be a bipartition of G, and let D be the orientation of G obtained by directing all edges from X to Y. We may assume that G has a perfect matching M, for otherwise the answer is clear. We may also assume that M is not even. By Exercise 3.5.5 the graph G has no even perfect matching if and only if D is Pfaffian.

Conversely, let G, D be a bipartite instance of IS ORIENTATION PFAFFIAN. We may assume that G has a perfect matching M, for otherwise D is Pfaffian. If

M is not even, then let $H := G$; otherwise let H be obtained from G by adding four vertices and two edges e_1, e_2 in such a way that F is a perfect matching in G if and only if $F \cup \{e_1, e_2\}$ is a perfect matching in H, and one is even if and only if the other one is not. Then by Exercise 3.5.5 D is a Pfaffian orientation of G if and only if H has no even perfect matching, as desired. □