MATH/ISyE/CS 7510: Graph Algorithms	Spring 2009

Lecture 3 — January 21

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The following result is due to Jack Edmonds.

Theorem 1. *Let G be a graph and let $\mathcal{M} \subseteq \mathbb{R}^{E(G)}$ denote the incidence vectors of all perfect matchings of G, then:*

$$conv(\mathcal{M}) = \{\bar{x} \in \mathbb{R}^{E(G)} : \ \bar{x} \ \textit{satisfies} \ (1), (2), (3)\}$$

$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in V(G) && (1) \\ x(e) &\geq 0 \quad \forall e \in E(G) && (2) \\ x(C) &\geq 1 \quad \forall \ \textit{odd cuts} \ C && (3) \end{aligned}$$

Proof. Let $P := \{\bar{x} \in \mathbb{R}^{E(G)} : \ \bar{x}$ satisfies $(1), (2), (3)\}$. Clearly $conv(\mathcal{M}) \subseteq P$, because all perfect matchings satisfy constraints $(1) - (3)$ and P is convex.

To show $P \subseteq conv(\mathcal{M})$, suppose there exists $x \in P - conv(\mathcal{M})$. Then $\exists \bar{w}$ and scalar t such that

$$\sum_{e \in E(G)} \bar{w}_e y_e \geq t \quad \forall y \in conv(\mathcal{M})$$

$$\text{and} \quad \sum_{e \in E(G)} \bar{w}_e x_e < t$$

Run the weighted matching algorithm on G using weight vector $\bar{w}$. Let y_v, Y_C be the dual solution and M be the corresponding matching found by the weighted matching algorithm. Then:

$$y_u + y_v + \sum_{C \ni e} Y_C = w_e \quad \forall e = uv \in M$$

$$\text{and} \quad Y_C \, 0 \quad \implies \quad |C \cap M| \geq 1.$$

From this we derive a contradiction:

$$t \quad > \quad \sum_{e \in E(G)} w_e x_e \geq \min_{z \in P} \sum_{e \in E(G)} w_e z_e = \sum_{e \in M} w_e \geq t$$

Thus $P \subseteq conv(\mathcal{M})$ which implies $P = conv(\mathcal{M})$. □

Definition: A graph G is said to be *matching covered* if it is connected and every edge belongs to a perfect matching.

Definition: $aff(\mathcal{M}) = \{\sum_{i=1}^{k} \lambda_i \bar{x}_i : \ \bar{x}_i \in \mathcal{M}, \sum_{i=1}^{k} \lambda_i = 1\}$.

Definition: $lin(\mathcal{M}) = \{\sum_{i=1}^{k} \lambda_i \bar{x}_i : \ \bar{x}_i \in \mathcal{M}\}$.

Definition: A cut C in a graph G is called a *tight cut* if $|C \cap M| = 1$ for every perfect matching M in G.

Theorem 2. *Let G be a matching covered graph, then*

$$aff(\mathcal{M}) = \{\bar{x} \in \mathbb{R}^{E(G)} : \ \bar{x}(C) = 1 \ \forall \ \textit{tight cuts } C\}.$$

Proof. Let $P = \{\bar{x} \in \mathbb{R}^{E(G)} : \ \bar{x}(C) = 1 \ \forall \text{ tight cuts } C\}$. Clearly $aff(\mathcal{M}) \subseteq P$ since all matchings, and affine combinations of matchings must satisfy $\bar{x}(C) = 1$ for all tight cuts C.

To show $P \subseteq aff(\mathcal{M})$, suppose $\bar{x}_1 \in P$, so $\bar{x}_1(C) = 1$ for all tight cuts C. Let $\bar{x}_2 = \frac{1}{|\mathcal{M}|} \sum_{\bar{x} \in \mathcal{M}} \bar{x}$ and let $\bar{x}_3 = \epsilon \bar{x}_1 + (1-\epsilon)\bar{x}_2$ for $\epsilon > 0$. If ϵ is sufficiently small then $\bar{x}_3 \in conv(\mathcal{M})$, and

$$\bar{x}_1 = \frac{1}{\epsilon}\bar{x}_3 - \left(\frac{1-\epsilon}{\epsilon}\right)\bar{x}_2 \in aff(\mathcal{M})$$

as desired. □

Corollary 3. *If G is a matching covered graph then:*

$$lin(\mathcal{M}) = \{\bar{x} \in \mathbb{R}^{E(G)} : \ \bar{x}(C) = \bar{x}(D) \quad \textit{for any two tight cuts } C, D\}.$$

Example 1. *Let G be a bipartite graph with bipartition (A, B) and let $X \subseteq A$ such that $|N(X)| = |X| + 1$, then $\delta(X \cup N(X))$ is a tight cut.*

Example 2. *Let G be a graph and $X \subseteq V(G)$ such that $|X| = o(G \backslash X)$. Let H be an odd component of $G \setminus X$ with at least three vertices, then $\delta(V(H))$ is a tight cut.*

Example 3. *Suppose G is not 3-connected and let $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = 2$, and $x \in V(G_1) \cap V(G_2)$. If $V(G_1), V(G_2)$ are odd, then $\delta(V(G_2))$ is a tight cut. If $V(G_1), V(G_2)$ are even, then $\delta(V(G_2) \setminus \{x\})$ is a tight cut.*

Exercise 1. *Find a tight cut in a matching covered graph that is not of any of the three forms in the preceding examples.*

Exercise 2. *Prove that in a matching covered bipartite graph, every tight cut is of the form described in Example 1.*

Definition: A *brace* is a bipartite matching covered graph on at least four vertices that has no tight cuts.

Exercise 3. *Let G be a bipartite matching covered graph on at least four vertices. Prove that G is a brace if and only if every matching of size two extends to a perfect matching.*

Definition: A *brick* is a 3-connected graph G such that $G \setminus \{u, v\}$ has a perfect matching for every two distinct vertices $u, v \in V(G)$.

Theorem 4 (Lovász)**.** *A matching covered graph on at least four vertices has no non-trivial tight cut if and only if it is a brace or a brick.*