Graph Algorithms Notes: Jan 26, 2009

Scribed by Carl Yerger

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1 Exercise Solutions

Exercise: Give an example of a tight cut that is not one of the three types discussed in class.
Solution:

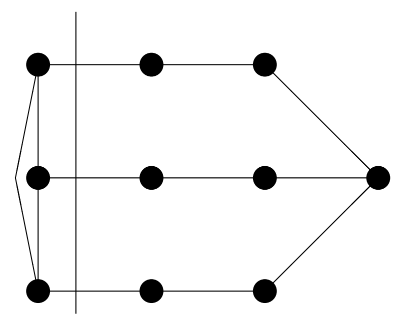

Figure 1: The vertical line is an example of such a cut.

Exercise: Show that every matching covered bipartite graph has cuts of the form (A, B) where $X \subseteq A$ such that $|X| = |N(X)| - 1$.
Solution: Suppose for purposes of contradiction this is not true. Take a tight cut, C. Since the cut is tight, any perfect matching, call it M, intersects the cut once. Let Y be the set of vertices on one side of the cut in one bipartition and let X be the vertices on the same side of the cut in the other bipartition. Without loss of generality let Y contain the vertex, say a, which is adjacent to a vertex on the other side of the cut in M. Notice that $|Y| = |X| + 1$. We must now show that Y is not the neighborhood of X.

First, consider the case when $a \in N(X)$. Then for us to not satisfy the hypothesis that $|X| = |N(X)| - 1$, there must be some vertex of X, say b, that is adjacent to a vertex, call it c, outside of Y. Since G is matching covered, consider a perfect matching that is forced to include edge bc. At this point there are $|X|+1$ unmatched vertices in Y and $|X|-1$ unmatched vertices in X. As a result in any perfect matching, there must be at least two vertices in Y that are incident to vertices outside of X. This contradicts the fact that the cut is tight.

Next, consider the case when $a \notin N(X)$. In this case since G is connected, there exists some edge with one end in either X or $Y \setminus \{a\}$ and one end outside the cut. Since G is matching covered, there exists a cut where this edge is included, and this crosses C. However, the edge of the perfect matching that includes vertex a crosses the cut, and thus our cut was not tight.

Exercise: Let G be a connected bipartite graph, let M be a perfect matching in G and let $k \geq 0$ be an integer. (Let (A, B) be a bipartition in G). Then prove that the following are equivalent:

1. Every matching of G of size at most k is contained in a perfect matching.

2. For every $X \subseteq A$, $|N(X)| \geq |X| + k$ or $N(X) = B$.

3. The digraph $D(G, M)$ is strongly k-connected.

Solution: First we will show (2) $\implies$ (1). Suppose not. Then take a matching of size at most k. Assume that it does not extend to a perfect matching. By Hall's theorem, there then exists a set whose neighborhood is not large enough, that is $|N(X)| < |X|$.

Next, we will show (2) $\implies$ (3). In $D(G, M)$, fix $|Y| \leq k-1$, $Y \subseteq V(D)$. Let U be the set of reachable vertices from $u \in V(D) - Y$. We claim that $U = V(G) - Y$, that is, the rest of the graph. Let U_A be the corresponding vertices to U in bipartition A from $G = (A, B)$. By hypothesis (2), since $|Y| \leq k-1$, there exists an edge from U to a vertex in $V(D) - U - Y$, a contradiction.

To show (3) $\implies$ (2) assume there exists some set X which violates the condition. We will use Hall's theorem. Suppose that $|N(X)| \leq |X| + k - 1$. Let Y be the set of vertices corresponding to $N(X) - M$. So Y is the set of at most $k-1$ unmatched vertices. Therefore, in $D(G, M)$ the graph should still be strongly connected after removing the vertices in Y. However, there is no dipath from $x \in X$ to any vertex corresponding to some other vertex b in D. This is a contradiction as G was assumed to be strongly k-connected.

Finally, to show (1) $\implies$ (2), we will use induction on k. If $k = 0$, we know that there exists a perfect matching. By Hall's condition, every set $X \in A$ has a neighborhood of size at least $|X|$. This satisfies condition (2). Now suppose that condition (1) holds up to $k-1$. So we know $|N(X)| \geq |X| + k - 1$. Let $C = X$, let set D equal the set of vertices in B not in $N(X)$, let E equal the set of vertices in B matched to X, let Z be the set of vertices in A matched to the (at least) $k-1$ vertices in $N(x)$ that are not matched to X. Let F be the set of remaining vertices in A. Condition (2) holds if there exists an edge from C to D or from E to F. In the first case $N(x) \geq |X| + k$. In the second case, we know that since G is assumed to be k-extendable, choosing the $k-1$ matched edges from Z as well as the edge from C to D to be the forced matched edges, forces $N(x) \geq |X| + k$.

Thus by connectivity of G, there exists some edge from D to Z or some edge from F to H, the set of $k-1$ vertices in $N(X)$ but not initially matched to X. First, suppose there exists an edge from D, at vertex d, to Z. From the original matching, remove the matching edge incident to vertex d and instead include this new edge from d to Z. Also, force a matching edge from C to the vertex in $N(X)$ which d was matched to. Also, match the remaining $k-2$ vertices in H to vertices outside X. This forces there to exist an edge between a vertex in E and a vertex in F, and we have already considered this case.

Now, suppose there exists an edge from F to H, say edge fh. If there exists some other vertex in F adjacent to a vertex in E, then we choose k matching edges as follows: edge fh, an edge from F to E, and the $(k-2)$ other edges that go from H to Z. This forces X to be adjacent to some vertex in E, so $|N(X)| \geq |X| + k$. Therefore we may assume there are no vertices in Z adjacent to E. Now since the vertices in H are in $N(X)$, choose some edge from X to H and include it in the matching. This forces an edge from E to F because there are no vertices in E that are adjacent to Z, and only $|X| - 1$ vertices can be matched to X. This gives a contradiction.

Exercise: Let M be a matching in G and let C be an M-blossom. Show that M is a maximum matching in G if and only if $M - E(C)$ is a maximum matching in $G/E(C)$ (contraction).

Solution: It suffices to show that G contains an M-augmenting path P if and only if $G/E(C)$ contains an $M - E(C)$ augmenting path Q.

First we will show the forward direction. Suppose that $P \cap E(C) = \emptyset$. In this case P will be a $M - E(C)$ augmenting path, a contradiction. Next, suppose that $P \cap E(C) \neq \emptyset$. We now must consider how an augmenting path and a blossom share edges. One end of the path could be a newly contracted vertex, but one is not. Choose the end that is not. Look at the nearest intersection point within the cycle of the path and follow this cycle to obtain the augmenting path.

Now, consider the opposite direction. Let v be the new vertex in $G/E(C)$ newly made by contraction. First suppose that $v \cap Q = \emptyset$. In this case the augmenting path will not change. Now if $v \cap Q \neq \emptyset$, then v is some end vertex, and we can perform the reverse process to make a blossom. This allows us to obtain an augmenting path in G.

2 Homework Project

Problem: Calculate the dimension of the linear hull of $\mathscr{M}(G)$, which describes the set of incidence vectors of perfect matchings of G over $\mathbf{GF}[2]$.

We will make the same computation in class but we will look at incidence vectors of perfect matchings over the real numbers. There are additional complications to the computation for $\mathbf{GF}[2]$. For instance, we will not be able to use Edmonds' polyhedral theorem.

Recall that $\mathscr{M}(G) = \{\lambda_1 \mathbb{1}_{M_1} + \lambda_2 \mathbb{1}_{M_2} + \cdots + \lambda_k \mathbb{1}_{M_k} : \lambda \in \mathbb{R}, M_i$ is a perfect matching $\}$. We showed

Theorem 1 *lin* $\mathscr{M}(G) = \{\bar{x} \in \mathbb{R}^{E(G)} : \bar{x}(C) = \bar{x}(D)$ for all tight cuts $C, D\}$

Corollary 2 *If G has no nontrivial tight cuts and is matching covered, then* $lin \mathscr{M}(G) = \{\bar{x} \in \mathbb{R}^{E(G)} : \bar{x}(\delta(u)) = \bar{x}(\delta(v))$for all $u, v \in G\}$.

Notice that $\text{lin}\mathscr{M}(G)$ comes remarkably close to the rank of the incidence matrix. So $\text{lin}\mathscr{M}(G) = \{\bar{x} \in \mathbb{R}^{E(G)} : A\bar{x} = \mathbf{X} = (dd \cdots d)^{\mathbf{T}}$for some $d\}$.

Let A be the incidence matrix:

$$\begin{pmatrix} v_1 e_1 & v_1 e_2 & \cdots & v_1 e_m \\ v_2 e_1 & v_2 e_2 & \cdots & v_2 e_m \\ \vdots & \vdots & \ddots & \vdots \\ v_n e_1 & v_n e_2 & \cdots & v_n 1 e_m \end{pmatrix} \begin{pmatrix} x e_1 \\ x e_2 \\ \vdots \\ x e_m \end{pmatrix}$$

So dim lin $\mathscr{M}(G) = m - \text{rank}(A) + 1$ for matching covered graphs with nontrivial tight cuts. However, we still wish to compute $\text{rank}(A)$.

Proposition 3 *If G is connected then*

$$rank(A) = \begin{cases} n & \text{otherwise} \\ n-1 & \text{if } G \text{ is bipartite} \end{cases} \quad (1)$$

Proof First, observe that each row in A can be viewed as a star, as this describes the incidences at a single vertex. So we have a linear combination of stars, $\lambda_1 \delta(v_1) + \lambda_2 \delta(v_2) + \cdots + \lambda_n \delta(v_n) = 0$. Notice, for every edge, $\lambda_i + \lambda_j = 0$. As a result, if $\lambda_i = A$ then all the neighbors of λ_i must equal $-A$. Also, the vertices distance two from vertex i must equal A, and so forth. The only way for the λ's to be anything other than zero is for G to be bipartite so that A and $-A$ can correspond to a 2-coloring of G. □

Therefore, we have shown:

Theorem 4 *If G is matching-covered, and has no nontrivial tight cut, then* $\text{dimlin}\mathscr{M}(G) = |E(G)| - |V(G)| + 2 - b(G)$, *where*

$$b(G) = \begin{cases} 1 & \text{if } G \text{ is a brick} \\ 0 & \text{if } G \text{ is a brace} \end{cases} \quad (2)$$

3 Tight Cut Decompositions

Suppose that G is a graph with a tight cut Γ, which divides G into G_1 and G_2. If G is matching covered, then G_1 obtained by collapsing all of G_2 into a single vertex, and G_2 obtained by collapsing all of G_1 into a single vertex are also matching covered. Here G_1 and G_2 are called *C-contractions*. In a *tight cut decomposition*, we start with G, take a tight cut and make C-contractions to create G_1 and G_2. If either of these has a non-trivial tight cut then we will repeat the process. At the end, we end up with a list of bricks and braces.

Definition 5 *Let $b(G)$ be defined as the number of bricks in the final list of a tight cut decomposition.*

It is not clear that this is well-defined. However, Lovasz showed that not only is $b(G)$ well-defined, but you get the same list of graphs, ignoring parallel edges. We will show in class that $b(G)$ is well-defined (but not Lovasz's result). To do this we will need to work out what happens when we get a tight cut.

4 Additional Hints for Project

Idea: Look at the orthogonal complement over **GF**[2]. Namely, study sets $X \subseteq E(G)$ such that $|X \cap M|$ is even for every perfect matching M.

Here are some examples. First consider even cuts. A perfect matching intersects it an even number of times.

Next, if n, the number of vertices is divisible by four, then consider the complement of an even cut, say $n = 4k$. Let M be a perfect matching. Then $|M \cap C|$ is even, say $2l$. Since there are $2l$ edges, there are $4l$ vertices. Since $n = 4k$, there are $4(k-l)$ vertices left to be matched by edges of X. Thus, there is an even number of vertices left. If n was not divisible by four, it still must be even, but now taking the complement of an odd cut gives you the same result.