CHAPTER 1

Matching theory

In this chapter we review the matching theory that will be needed later on.

1.1. Matchings and matching decomposition

All *graphs* in this book are finite, may have loops and parallel edges and are undirected. Similarly, *directed graphs* or *digraphs* may have loops and parallel edges. Most of our terminology is standard and can be found in many textbooks, such as [**5, 16, 84**]. In particular, *cycles* and *paths* have no repeated vertices. A *matching* in a graph G is a set $M \subseteq E(G)$ such that every vertex of G is incident with at most one edge of G. We say that M *saturates* a vertex $v \in V(G)$ if v is incident with an edge of M. A matching M in a graph G is a *perfect matching* if it saturates every vertex of G. The following two classical theorems characterize graphs without perfect matchings; their proofs can be found in almost any graph theory texbook. The first is due to Hall [**26**] and the second is due to Tutte [**80**]. For a graph G and $X \subseteq V(G)$ we denote by $N(X)$ the set of vertices of $V(G) - X$ that have a neighbor in X.

Theorem 1.1.1. *Let G be a bipartite graph with bipartition (A, B). Then G has a matching saturating every vertex of A if and only if $|N(X)| \geq |X|$ for every $X \subseteq A$.*

Theorem 1.1.2. *A graph G has a perfect matching if and only if for every set $X \subseteq V(G)$ the graph $G\backslash X$ has at most $|X|$ components with odd number of vertices.*

In many matching-related problems edges that do not belong to any perfect matching are irrelevant and may be deleted. That motivates the following definitions. Let G be a graph, and let $k \geq 0$ be an integer. We say that G is *k-extendable* if every matching $M \subseteq E(G)$ of size at most k is contained in a perfect matching of G. A 1-extendable connected graph is called *matching-covered*. Matching covered bipartite graphs are closely related to strongly connected graph by means of an elementary, but important, construction. A digraph D is *strongly connected* if for every two vertices u and v it has a directed path from u to v. It is *strongly k-connected*, where $k \geq 1$ is an integer, if for every set $X \subseteq V(D)$ of size less than k, the digraph $D\backslash X$ is strongly connected. Let G be a bipartite graph with bipartition (A, B), and let M be a perfect matching in G. We define $D = D(G, M)$ to be the digraph obtained from G by directing every edge from A to B, and contracting every edge of M.

Exercise 1.1.3. Let G be a connected bipartite graph with bipartition (A, B), let M be a perfect matching in G, let $D := D(G, M)$, and let $k \geq 0$ be an integer. Then the following conditions are equivalent:
(1) G is k-extendable
(2) for every set $X \subseteq A$ either $N(X) = B$ or $|N(X)| \geq |X| + k$,

(3) D is strongly k-connected.

Exercise 1.1.4. Define k-extendability of bipartite graphs for $k < 0$ so that (1) and (2) of Exercise 1.1.3 remain equivalent.

There is an ear-decomposition of matching-covered graphs, an analogue of the better known ear-decomposition of strongly connected graphs, the following.

Exercise 1.1.5. Prove that a bipartite graph G is matching-covered if and only if it can be written as $G = G_0 \cup G_1 \cup \cdots \cup G_k$, where $k \geq 0$ is an integer, G_0 is isomorphic to K_2, and for $i = 1, 2, \ldots, k$ the graph G_i is an odd path with both ends in $G_0 \cup G_1 \cup \cdots \cup G_{i-1}$ and otherwise disjoint from it.

Exercise 1.1.6. Let G be a matching-covered bipartite graph with bipartition (A, B), and let M be a perfect matching in G. Prove that for every $a \in A$ and every $b \in B$ there exists an M-alternating path P in G with ends a and b such that the first and last edge of P belong to M.

Let G be a graph, and let $X \subseteq V(G)$. We use $\delta(X)$ to denote the set of edges with one end in X and the other in $V(G) - X$. A *cut in a graph* G is any set of edges of the form $\delta(X)$ for some $X \subseteq V(G)$. The sets X and $V(G) - X$ are called the *shores of the cut* $\delta(X)$. A cut C is a *tight cut* if $|C \cap M| = 1$ for every perfect matching M in G. Every cut of the form $\delta(\{v\})$ in a graph with a perfect matching is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*.

Exercise 1.1.7. Prove that the shores of a tight cut in a matching covered-graph G induce connected subgraphs of G.

Let G be a graph. A set $X \subseteq V(G)$ is called a *barrier* in G if $G \backslash X$ has exactly $|X|$ odd components. The following exercises give important examples of tight cuts.

Exercise 1.1.8. Let G be a graph, let $X \subseteq V(G)$ be a barrier in G, and let C be an odd component of $G \backslash X$. Then $\delta(V(C))$ is a tight cut.

Exercise 1.1.9. Let G be a matching-covered graph, let u, v be distinct vertices of G such that $G \backslash \{u, v\}$ is disconnected, and let A be the vertex-set of a component of $G \backslash \{u, v\}$. Prove that if $|A|$ is even, then $\delta(A \cup \{u\})$ is a nontrivial tight cut. (Notice that if $|A|$ is odd, then $\delta(A)$ is a tight cut of the form described in the previous exercise).

We say a tight cut is a *barrier cut* if it is of the form described in Exercise 1.1.8, and we say that it is a *2-separation cut* if it is of the form described in Exercise 1.1.9.

Exercise 1.1.10. Let G be a a bipartite matching-covered graph with bipartition (A, B). Prove that every nontrivial tight cut is of the form $\delta(X \cup N(X))$, where $X \subseteq A$ and $|N(X)| = |X| + 1 < |A|$. Deduce that every nontrivial tight cut in G is a barrier cut.

Exercise 1.1.11. Find a matching-covered graph G and a nontrivial tight cut C in G that is not a barrier cut or a 2-separation cut.

On the other hand, Theorem 1.1.12 below implies that if a graph has a nontrivial tight cut, then it has a nontrivial barrier or 2-separation cut. A *brick* is a 3-connected graph G such that $G \backslash \{u, v\}$ has a perfect matching for every two distinct vertices u, v of G. A *brace* is a bipartite graph such that every matching of size at most two is a subset of a perfect matching. The following result of Edmonds, Lovász and Pulleyblank [**17, 18**] characterizes graphs with no notrivial tight cut.

Theorem 1.1.12. *Let G be a matching covered graph. Then G has no nontrivial tight cut if and only if G is a brick or a brace.*

Exercise 1.1.13. For every matching-covered bipartite graph G with bipartition (A, B) the following conditions are equivalent:

(i) G is a brace,

(ii) G has no tight cut,

(iii) for every set $X \subseteq A$ either $N(X) = B$ or $|N(X)| \geq |X| + 2$,

(iv) for every four distinct vertices $a, a' \in A$, $b, b' \in B$ the graph $G\backslash\{a, a', b, b'\}$ has a perfect matching.

Let $\delta(X)$ be a cut in a graph G. Let G_1 be obtained from G by identifying all vertices in X into a single vertex, and let G_2 be defined analogously by identifying all vertices in $V(G) - X$. We say that G_1 and G_2 are the two *C-contractions* of G. Let us clarify a technical point here. We assume that $E(G_1) \subseteq E(G)$; in other words, during the contraction we keep the same edges, but change the incidences of some. In particular, the operation of C-contraction may create parallel edges.

Exercise 1.1.14. Let G be a matching-covered graph, let C be a tight cut in G, and let G_1 and G_2 be the two C-contractions. Let $X \subseteq V(G) - V(G_2)$. Then $\delta(X)$ is a cut in both G and G_1, and it is a tight cut in G if and only if it is a tight cut in G_1.

It follows from Excercises 1.1.7 and 1.1.14 that if G is matching-covered, then so are G_1 and G_2. Many matching-related problems can be solved for G if we are given the corresponding solutions for G_1 and G_2.

Let us now apply this construction recursively, as follows. Let G be a matching-covered graph, and let initially $\mathcal{L} = \{G\}$. If some graph H in $\mathcal{L}$ has a nontrivial tight cut C, then we replace H by the two C-contractions of H, and continue this process until $\mathcal{L}$ consists entirely of graphs with no nontrivial tight cut. By Theorem 1.1.12 the members of $\mathcal{L}$ are bricks and braces. Let the list $\mathcal{L}'$ consist of the underlying simple graphs of members of $\mathcal{L}$, each graph listed as many times as it appears in the list. We say that the members of $\mathcal{L}'$ are the *bricks* and *braces* of G. Lovász [**44**] proved the following.

Theorem 1.1.15. *For every matching-covered graph, the list of bricks and braces of G is, up to isomorphism, independent of the choice of tight cuts during the decomposition.*

Thus for a matching-covered graph G we define $\mathrm{b}(G)$ to be the number of bricks in the list of bricks and braces of G, and we define $\mathrm{p}(G)$ to be the number of those bricks of G isomorphic to the Petersen graph. In particular, we say that a *graph G has at most one brick* if $\mathrm{b}(G) \leq 1$.

The process of arriving at the graphs in $\mathcal{L}$ is usually called a tight cut decomposition of G. However, for later reference we prefer an equivalent definition, as follows. We say that two cuts $\delta(X)$ and $\delta(Y)$ in a graph G *cross* if each of the four sets $X \cap Y$, $X - Y$, $Y - X$, and $V(G) - X - Y$ is non-empty. A family $\mathcal{F}$ of cuts in G is called *laminar* if no two members of $\mathcal{F}$ cross. Each of the tight cuts used in process of obtaining $\mathcal{L}$ is a tight cut in G by Exercise 1.1.14; let $\mathcal{F}$ denote the set of all those tight cuts in G. Then $\mathcal{F}$ is a maximal laminar family of nontrivial tight cuts in G, and, conversely, every maximal laminar family of nontrivial tight cuts gives rise to a list $\mathcal{L}$ as above. In light of these remarks we define a *tight cut* *decomposition of a graph* G to be any maximal laminar family of nontrivial tight cuts in G. By Theorem 1.1.15 every two tight cut decompositions of the same graph have the same cardinality.

1.2. Building bricks

The following theorem was conjectured by Lovász and proved by de Carvalho, Lucchesi and Murty [**12, 13**]. A proof was also announced by Lovász and Vempala, but has not yet appeared. By a *prism* we mean the unique 3-regular planar graph on six vertices.

Theorem 1.2.1. *Every brick* G *other than* K_4*, the prism and the Petersen graph has an edge* e *such that* $G\backslash e$ *is a matching-covered graph with at most one brick, not isomorphic to the Petersen graph.*

Theorem 1.2.1 is very useful, and therefore we give a proof, even though it is quite long and technical. In fact, we will prove a stronger statement, also due to de Carvalho, Lucchesi and Murty [**14**]. If v is a vertex of degree two in a graph G adjacent to two distinct neighbors, and H is the graph obtained from G by contracting both edges incident with v, then we say that H was obtained from G by *bicontracting the vertex* v. We say that a graph H is a *retract of a graph* G if H is obtained from G by repeatedly bicontracting all vertices of degree two.

Theorem 1.2.2. *Every brick* G *other than* K_4*, the prism and the Petersen graph has an edge* e *such that the retract of* $G\backslash e$ *is a brick not isomorphic to the Petersen graph.*

