10. PLANAR GRAPHS CONTINUED

10A. SEPARATORS

A1. Exercise. Prove that for every tree T on n vertices there exists a separation (T_1, T_2) of T of order at most one such that $|V(T_1)|, |V(T_2)| > \frac{1}{3}\, n$.

A2. Exercise. Prove that for every series–parallel graph G on n vertices there exists a separation (G_1, G_2) of G of order at most two such that $|V(G_1)|, |V(G_2)| > \frac{1}{3}\, n$.

A3. Definition. A planar drawing Γ is a *triangulation* if every face of Γ is bounded by a circuit of length three. A *near-triangulation* is a planar drawing such that every face, except possibly the unbounded one, is bounded by a circuit of length three.

A4. Lemma. *Let Γ be a simple near-triangulation with the unbounded face bounded by a circuit C. Let u and v be two distinct vertices of C, let P_1 and P_2 be the two subpaths of C with ends u and v and union C, and let S be a set of vertices of Γ with $u, v \in S$. Then either there exists a path P in Γ with ends u and v satisfying $V(P) \subseteq S$, or there exists a path Q in Γ between $V(P_1)$ and $V(P_2)$ with $V(Q) \cap S = \emptyset$.*

Proof. We proceed by induction on $V(\Gamma)$. We may assume that P_1 does not satisfy the conclusion of the lemma, and hence there exists a vertex $w \in V(P_1) - S$. First let us assume that some vertex $w' \in V(C)$ is adjacent to w in Γ, but not in C. Then Γ has a 2-separation (Γ_1, Γ_2) such that $V(\Gamma_1) \cap V(\Gamma_2) = \{w, w'\}$, $V(\Gamma_1) - V(\Gamma_2) \neq \emptyset$ and $V(\Gamma_2) - V(\Gamma_1) \neq \emptyset$. If $w' \in V(P_1)$, then one of Γ_1, Γ_2 contains P_2, say Γ_2. The lemma then follows from the induction hypothesis applied to Γ_2. Thus we may assume that $w' \notin V(P_1)$. We may also assume that $w' \in S$, for otherwise the path with vertex-set $\{w, w'\}$ satisfies the conclusion of the lemma. Finally, we may assume that the notation is chosen so that $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$. We apply the induction hypothesis to the triples Γ_1, u, w' and Γ_2, v, w'. The lemma follows by suitably combining the resulting paths.

We have thus shown that if C has a vertex w' as above, then the lemma holds. We may therefore assume that no such vertex exists. Since Γ is a simple near-triangulation, the neighbors of w induce a path R in Γ. By the nonexistence of a vertex w' as above we deduce that R is disjoint from C, except for its ends. Let $\Gamma' = \Gamma \backslash w$; then Γ' is a near-triangulation with the unbounded region bounded by the circuit $R \cup C \backslash w$. The lemma now follows easily from the induction hypothesis applied to Γ'. $\square$

A5. Lemma. *Let Γ be a near-triangulation with the infinite face bounded by a circuit C. Let the vertices of C be $v_0, v_1, \ldots, v_t = v_0$ (in order), let i be an integer with $0 < i < t$, and let k be a positive integer. Then either Γ has k pairwise disjoint paths between $\{v_0, v_1, \ldots, v_i\}$ and $\{v_i, v_{i+1}, \ldots, v_t\}$, or there exists a path P in Γ between v_0 and v_i with $|V(P)| < k$.*

Proof. By Menger's theorem there either exist the k disjoint paths as in the statement of the lemma, or there exists a set $S \subseteq V(\Gamma)$ such that $|S| < k$ and there is no path between $\{v_0, v_1, \ldots, v_i\}$ and $\{v_i, v_{i+1}, \ldots, v_t\}$ in $\Gamma \backslash S$. By 10A4 there exists a path P between v_0 and v_i with $V(P) \subseteq S$. Thus $|V(P)| < k$, as desired. □

A6. Theorem. (Lipton, Tarjan [5]) *For every planar graph on n vertices there exists a separation (G_1, G_2) of G of order at most $2\sqrt{2}\sqrt{n}$ such that $|V(G_1)|, |V(G_2)| > \frac{1}{3}\,n$.*

Proof. We follow [2]. We may assume that G is a planar drawing, that it has no loops or multiple edges, that $n \geq 3$ and (by adding new edges to G) that G is a triangulation. Let $k = \lfloor\sqrt{2n}\rfloor$. For any circuit C of G we denote by $A(C)$ and $B(C)$ the sets of vertices drawn inside C and outside C, respectively; thus $(A(C), B(C), V(C))$ is a partition of $V(G)$, and no vertex in $A(C)$ is adjacent to any in $B(C)$. Choose a circuit C of G such that

(i) $|V(C)| \leq 2k$
(ii) $|B(C)| < \frac{2}{3}\,n$
(iii) subject to (i) and (ii), $|A(C)| - |B(C)|$ is minimum.

This is possible, because the circuit bounding the infinite region satisfies (i) and (ii). Let G_1 be the subgraph of G induced by $A(C) \cup C$, and let G_2 be the subgraph of G induced by $B(C) \cup C$. We claim that (G_1, G_2) satisfies the conclusion of the theorem. We suppose, for a contradiction, that that is not the case; then $|A(C)| \geq \frac{2}{3}\,n$. Let D be the subgraph of G drawn in the closed disc bounded by C. For $u, v \in V(C)$, let $c(u, v)$ (respectively, $d(u, v)$) be the number of edges in the shortest path of C (respectively, D) between u and v.

(1) *$c(u, v) = d(u, v)$ for all $u, v \in V(C)$.*

For certainly $d(u, v) \leq c(u, v)$ since C is a subgraph of D. If possible, choose a pair $u, v \in V(C)$ with $d(u, v)$ minimum such that $d(u, v) < c(u, v)$. Let P be a path of D between u and v, with $d(u, v)$ edges. Suppose that some internal vertex w of P belongs to $V(C)$. Then

$$d(u, w) + d(w, v) = d(u, v) < c(u, v) \leq c(u, w) + c(w, v)$$

and so either $d(u, w) < c(u, w)$ or $d(w, v) < c(w, v)$, in either case contrary to the choice of u, v. Thus there is no such w. Let C, C_1, C_2 be the three circuits of $C \cup P$ where $|A(C_1)| \geq |A(C_2)|$. Now $|B(C_1)| < \frac{2}{3}\,n$, since

$$n - |B(C_1)| = |A(C_1)| + |V(C_1)| > \frac{1}{2}\left(|A(C_1)| + |A(C_2)| + |V(P)| - 2\right) = \frac{1}{2}\,|A(C)| \geq \frac{1}{3}\,n.$$

But $|V(C_1)| \leq |V(C)|$ since $|E(P)| \leq c(u, v)$, and so C_1 satisfies (i) and (ii). By (iii), $B(C_1) = B(C)$, and in particular $c(u, v) \leq 1$, which is impossible since $d(u, v) < c(u, v)$. This proves (1).

(2) $|V(C)| = 2k$.

For suppose that $|V(C)| < 2k$. Choose $e \in E(C)$, and let P be the two-edge path of D such that the union of P and e forms a circuit bounding a region inside of C. Let v be the middle vertex of P, and let P' be the path $C \backslash e$. Now $P \neq P'$ since $A(C) \neq \emptyset$, and so $v \notin V(C)$ by (1). Hence $P \cup P'$ is a circuit satisfying (i) and (ii), contrary to (iii). This proves (2).

Let the vertices of C be $v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0$, in order.

(3) *There are $k+1$ vertex-disjoint paths of D between $\{v_0, v_1, \ldots, v_k\}$ and $\{v_k, v_{k+1}, \ldots, v_{2k}\}$.*

Indeed, otherwise, by the previous lemma, there is a path of D between v_0 and v_k with less than k vertices, contrary to (1). This proves (3).

Let the paths of (3) be $P_0, P_1, \ldots, P_k$, where P_i has ends v_i, v_{2k-i} $(0 \leq i \leq k)$. By (1),

$$|V(P_i)| \geq \min(2i+1, 2(k-i)+1)$$

and so

$$n = |V(G)| \geq \sum_{0 \leq i \leq k} \min(2i+1, 2(k-i)+1) \geq \frac{1}{2}\,(k+1)^2.$$

Yet $k+1 > \sqrt{2n}$ by the definition of k, a contradiction. Thus our assumption that $|A(C)| \geq \frac{2}{3}\,n$ was false, and so $|A(C)| < \frac{2}{3}\,n$ and (G_1, G_2) satisfies the theorem. ☐

A7. Remark. Lipton and Tarjan [5] gave a linear time algorithm to find a separation as in 10A6. The proof we presented gives a quadratic algorithm.

A8. Remark. It is not known what is the best constant c that can replace $2\sqrt{2}$ in 10A6. It is known [2, 4] that $1.555 \doteq \frac{1}{3}\sqrt{4\pi\sqrt{3}} \leq c \leq \frac{3}{2}\sqrt{2}$.

A9. Remark. There are several applications of 10A6 [6, 7]. Here is one. Notice that planarity is only used to find a separation as in 10A6.

A10. Proposition. *There exists an algorithm that finds the size of a maximum independent set of a planar graph on n vertices in time $2^{O(\sqrt{n})}$.*

Proof. Let G be a planar graph, and let $F \subseteq Z \subseteq V(G)$. We denote by $\alpha(G, Z, F)$ the size of the maximum independent set I in G such that $I \cap Z = F$. For a separation (A, B) of G let $C = V(A) \cap V(B)$; then

$$\alpha(G, Z, F) = \max\left\{\alpha(A, Z \cup C, F \cup X) + \alpha(B, Z \cup C, F \cup X) - |X| - |F \cap C|\right\},$$

where the maximum is taken over all sets $X \subseteq V(A) \cap V(B) - Z$. Using this formula and 10A6 recursively gives an algorithm to compute $\alpha(G, Z, F)$, whose worst case running time $f(n)$ satisfies the recursion

$$f(n) \leq O(n^2) + 2^{O(\sqrt{n})} \max(f(n_1) + f(n_2)),$$

where the maximum is taken over all integers n_1, n_2 with $n_1, n_2 > \frac{1}{3}n$ and $n_1 + n_2 \le n + 2\sqrt{2}\sqrt{n}$. It follows that $f(n) = 2^{O(\sqrt{n})}$. $\square$

A11. Remark. Our next objective is to prove a separator theorem for graphs with an excluded minor. Recall that X-flaps and havens are defined in the tree-width chapter.

A12. Lemma. [1] *Let G be a graph with n vertices, let $A_1, \ldots, A_k \subseteq V(G)$, and let r be a real number with $r \ge 1$. Then either*

(i) there is a tree T in G with $|V(T)| \le r$ such that $V(T) \cap A_i \ne \emptyset$ for $i = 1, \ldots, k$, or

(ii) there exists $Z \subseteq V(G)$ with $|Z| \le (k-1)n/r$, such that no Z-flap intersects all of $A_1, \ldots, A_k$.

Proof. We may assume that $k \ge 2$. Let $G^1, \ldots, G^{k-1}$ be isomorphic copies of G, mutually disjoint. For each $v \in V(G)$ and $1 \le i \le k-1$, let v^i be the corresponding vertex of G^i. Let J be the graph obtained from $G^1 \cup \cdots \cup G^{k-1}$ by adding, for $2 \le i \le k-1$ and all $v \in A_i$, an edge joining v^{i-1} and v^i. Let $X = \{v^1 : v \in A_1\}$ and $Y = \{v^{k-1} : v \in A_k\}$. For each $u \in V(J)$, let $d(u)$ be the number of vertices in the shortest path of J between X and u (or ∞ if there is no such path). There are two cases:

Case 1. $d(u) \le r$ for some $u \in Y$.

Let P be a path of J between X and Y with $\le r$ vertices. Let

$$S = \{v \in V(G) : v^i \in V(P) \text{ for some } i \text{ with } 1 \le i \le k-1\}.$$

Then $|S| \le |V(P)| \le r$, the subgraph of G induced on S is connected, and $|S \cap A_i| \ne \emptyset$ for $1 \le i \le k$. Thus (i) holds.

Case 2. $d(u) > r$ for all $u \in Y$.

Let t be the least integer with $t \ge r$. For $1 \le j \le t$, let $Z_j = \{u \in V(J) : d(u) = j\}$. Since $|V(J)| = (k-1)n$ and $Z_1, \ldots, Z_t$ are mutually disjoint, one of them, say Z_j, has cardinality at most $(k-1)n/t \le (k-1)n/r$. Now every path of J between X and Y has a vertex in Z_j, because $d(u) \ge j$ for all $u \in Y$. Let

$$Z = \{v \in V(G) : v^i \in Z_j \text{ for some } i \text{ with } 1 \le i \le k-1\}.$$

Then $|Z| \le |Z_j| \le (k-1)n/r$, and we claim that Z satisfies (ii). Suppose that F is a Z-flap of G which intersects all of $A_1, \ldots, A_k$. Let $a_i \in F \cap A_i$ $(1 \le i \le k)$, and for $1 \le i \le k-1$ let P_i be a path of G with $V(P_i) \subseteq F$ and with ends a_i, a_{i+1}. Let P^i be the path of G^i corresponding to P_i. Then $V(P^1) \cup \cdots \cup V(P^{k-1})$ includes the vertex set of a path of J between X and Y, and yet is disjoint from Z_j, a contradiction. Thus, there is no such F, and so (ii) holds. $\square$

A13. Open problem. Can the bound $(k-1)n/r$ in the lemma be improved to $o(k)n/r$? That would imply a corresponding improvement in 10A16 below.

A14. Definition. A *cluster* in a graph G is a set $\mathcal{C}$ of vertex-disjoint trees in G such that for every two distinct members $T, T' \in \mathcal{C}$ there exists an edge of G with one end in $V(T)$ and the other end in $V(T')$. Thus if G has a cluster of cardinality h, then G has a K_h-minor.

A15. Theorem. [1] *Let $h \geq 1$ be an integer and let G be a graph with n vertices and with a haven of order $h^{3/2}n^{1/2} + 1$. Then G has a K_h-minor.*

Proof. Let β be a haven in G of order $h^{3/2}n^{1/2}+1$. Choose $X \subseteq V(G)$ and a cluster $\mathcal{C}$ with $|\mathcal{C}| \leq h$ such that

(i) $X \subseteq \bigcup(V(C) : C \in \mathcal{C})$,
(ii) $|X \cap V(C)| \leq h^{1/2}n^{1/2}$ for each $C \in \mathcal{C}$,
(iii) $V(C) \cap \beta(X) = \emptyset$ for each $C \in \mathcal{C}$, and
(iv) subject to (i), (ii), and (iii), $|\mathcal{C}| + |X| + 3|\beta(X)|$ is minimum.

This is possible, because setting $\mathcal{C} = X = \emptyset$ satisfies (i), (ii), and (iii). Let $\mathcal{C} = \{C_1, \dots, C_k\}$. We suppose for a contradiction that $k < h$. For $1 \leq i \leq k$, let A_i be the set of all $v \in \beta(X)$ adjacent in G to a vertex of C_i. Let G' be the restriction of G to $\beta(X)$. By 10A12 applied to G' with $r = h^{1/2}n^{1/2}$, one of the following cases holds:

Case 1. There is a tree T of G' with $|V(T)| \leq h^{1/2}n^{1/2}$, such that $V(T) \cap A_i \neq \emptyset$ for $1 \leq i \leq k$. Let $\mathcal{C}' = \mathcal{C} \cup \{T\}$ and $X' = X \cup V(T)$; then $\mathcal{C}'$ is a cluster and for each $C \in \mathcal{C}'$,

$$V(C) \cap \beta(X') \subseteq V(C) \cap (\beta(X) - V(T)) = \emptyset.$$

This contradicts (iv).

Case 2. There exists $Z \subseteq \beta(X)$ with $|Z| \leq (k-1)|\beta(X)|/h^{1/2}n^{1/2} \leq h^{1/2}n^{1/2}$ such that no Z-flap of G' intersects all of $A_1, \dots, A_k$. Let $Y = X \cup Z$. Since $k \leq h-1$, it follows that $|Y| \leq h^{3/2}n^{1/2}$, and so $\beta(Y)$ exists and $\beta(Y) \subseteq \beta(X)$. Since $\beta(Y)$ is a Z-flap of G' there exists i with $1 \leq i \leq k$ such that $\beta(Y) \cap A_i = \emptyset$. Extend C_i to a maximal tree C_i' of G disjoint from $\beta(Y)$ and from each C_j $(j \neq i)$. Let $Z' = V(C_i') \cap Z$, let $X' = Z' \cup (X - V(C_i))$, and let $W = V(C_i') \cup (V(G) - \beta(X))$.

We claim that $\beta(X') \cap W = \emptyset$. For suppose not. Since $\beta(Y) \subseteq \beta(X')$, there is a path of G between W and $\beta(Y)$ contained within $\beta(X')$ and hence disjoint from X'. Since $W \cap \beta(Y) = \emptyset$, there are two consecutive vertices u, v of this path with $u \in W$ and $v \in V(G) - W \subseteq \beta(X)$. Since u, v are adjacent it follows that $u \in X \cup \beta(X)$, and so

$$u \in (X \cup \beta(X)) \cap (W - X') \subseteq V(C_i').$$

Since $v \notin W$ it follows from the maximality of C_i' that $v \in \beta(Y)$. Since $u \notin \beta(Y)$ we deduce that $u \in Y$, and so

$$u \in Y \cap (V(C_i') - X') \subseteq V(C_i).$$

But then $v \in A_i$, which is impossible since $A_i \cap \beta(Y) = \emptyset$. This proves our claim that $\beta(X') \cap W = \emptyset$. Hence, $\beta(X') \subseteq \beta(X)$. Let $\mathcal{C}' = (\mathcal{C} - \{C_i\}) \cup \{C_i'\}$; then $\mathcal{C}'$ is a cluster. We observe that

(i) $X' \subseteq \bigcup(V(C) : C \in \mathcal{C}')$; for $Z' \subseteq V(C_i')$,

(ii) $|X' \cap V(C)| \leq h^{1/2}n^{1/2}$ for each $C \in \mathcal{C}'$; for if $C \neq C_i'$ then $X' \cap V(C) = X \cap V(C)$, and $X' \cap V(C_i') = Z'$, and

(iii) $V(C) \cap \beta(X') = \emptyset$ for each $C = \mathcal{C}'$; for $\beta(X') \cap W = \emptyset$, as we have seen.

By (iv),

$$|\mathcal{C}'| + |X'| + 3|\beta(X')| \geq |\mathcal{C}| + |X| + 3|\beta(X)|.$$

But $|\mathcal{C}'| = |\mathcal{C}|$ and $X' \cup \beta(X') \subseteq (X \cup \beta(X)) - (X \cap V(C_i))$, and so $X \cap V(C_i) = \emptyset$. Then $\mathcal{C} - \{C_i\}$, X satisfy (i), (ii), and (iii), contrary to (iv).

In both cases, therefore, we have obtained a contradiction. Thus our assumption that $k < h$ was incorrect, and so $k = h$ and G has a K_h-minor, as required. □

A16. Theorem. [1] *Let h be an integer, and let G be a graph on n vertices with no K_h-minor. Then G has a separation (G_1, G_2) of order at most $h^{3/2}n^{1/2}$ such that $|V(G_1)|, |V(G_2)| \geq n/3$.*

Proof. Suppose that G does not have a separation as stated in the theorem. Then for every set $X \subseteq V(G)$ with $|X| \leq h^{3/2}n^{1/2}$ some component of $G\backslash X$ has more than $n/2$ vertices (exercise). Define $\beta(X)$ to be that component. Then β is a haven of order $h^{3/2}n^{1/2}+1$, and hence the theorem follows from 10A15. □

A17. Open problem. Can the bound $h^{3/2}n^{1/2}$ be improved to $O(hn^{1/2})$? That would be best possible up to a constant factor. See also 10A13.

A18. Remark. Bui, Fukuyama and Jones [3] have shown that that it is NP-hard to determine, given a planar graph G, the smallest integer k such that G has a separation (G_1, G_2) of order k satisfying $|V(G_1)|, |V(G_2)| \geq |V(G)|/3$.

10B. CHARACTERIZATIONS OF PLANARITY

B1. Exercise. If G is a planar graph, then the following are equivalent:

(i) G is a block not isomorphic to K_0, K_1, or K_2,

(ii) there exists a planar drawing of G such that every face is bounded by a circuit,

(iii) in every planar drawing of G, every face is bounded by a circuit.

B2. Definition. A graph G' is called an *abstract dual* of a graph G if $\mathcal{M}(G')$ is isomorphic to $\mathcal{M}^*(G)$.

B3. Exercise. Find a graph G such that some graph is an abstract dual of G, but not a geometric dual.