CHAPTER 1

Series-Parallel Graphs

Series-parallel graphs are a useful class of graphs—they are fairly simple and reasonably well-understood to allow easy proofs for many results, and at the same time they are rich enough so that many problems are non-trivial even when restricted to this class. In particular, series-parallel graphs are a fertile testing ground for various conjectures. We will use series-parallel graph as a first example of graph structure and the corresponding decomposition.

1.1. Preliminaries

Graphs are undirected, may have loops and multiple edges, and are finite, unless stated otherwise. More precisely, a *graph* G consists of a set of vertices $V(G)$, a set of edges $E(G)$ and a set of incidences between vertices and edges. Every edge is incident with two (not necessarily distinct) vertices, called its *ends*. If its ends are equal it is called a *loop*, otherwise it is called a *link*. Two edges are *parallel* if they have the same ends. A graph is *simple* if it has no loops and no parallel edges. *Paths* and *cycles* have no repeated vertices or edges, and are regarded as graphs. The graph with no vertices is called the *null graph* and is denoted by K_0. The null graph is (by definition) neither connected nor disconnected, and has no components. For an integer $n \geq 1$ we denote by K_n the complete graph on n vertices. If G is a graph and X is a vertex, a set of vertices, an edge, or a set of edges, then $G \backslash X$ denotes the graph that results when X is deleted from G. If G_1, G_2 are graphs then $G_1 \cup G_2$ is the graph with vertex-set $V(G_1) \cup V(G_2)$, edge-set $E(G_1) \cup E(G_2)$ and the obvious incidences. The graph $G_1 \cap G_2$ is defined similarly. A *separation* in a graph is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$. The *order* of (A, B) is $|A \cap B|$. If G is a graph and $X \subseteq V(G)$ we denote by $\delta(X)$ the set of edges with one end in X and the other in $V(G) - X$. A set $E \subseteq E(G)$ is called a *cut* if $E \neq \emptyset$ and $E = \delta(X)$ for some $X \subseteq V(G)$. (This is not the same as saying that $G \backslash E$ is disconnected.) An inclusion-wise minimal cut is called a *bond* of G. It is easy to see that every cut is a disjoint union of bonds. If v is a vertex of a graph, then the *degree*, denoted by $d_G(v)$ or $d(v)$, is the number of edges of G incident with v, counting loops twice. A *tree* is a connected graph with no cycles. A *forest* is a graph with no cycles. We say that two paths P, Q are *internally disjoint* if every vertex common to P and Q is an end of both.

Exercise 1.1.1. A cut $\delta(X)$ in a connected graph G is a bond if and only if $G \backslash X$ and $G \backslash (V(G) - X)$ are both connected.

Exercise 1.1.2. Let $T_1, T_2, \ldots, T_k$ be subtrees of a tree such that every two of them have a vertex in common. Prove that they all have a vertex in common.

Exercise 1.1.3. Prove that if we choose a direction for every edge of a tree T, then for some vertex t of T all the edges incident with t will be directed toward t.

1.2. Subdivisions and minors

In this section we define subdivisions and minors of graphs, and state a few straightforward exercises to illustrate these concepts.

Definition 1.2.1. Let G be a graph, and let $e \in E(G)$. As explained earlier, $G\backslash e$ denotes the graph obtained from G by *deleting* e. By G/e we denote the graph obtained from G by *contracting* e, where contraction is defined as follows. If e is a loop then G/e is defined to be $G\backslash e$; otherwise G/e is obtained by deleting e and identifying the ends of e. Thus $|E(G/e)| = |E(G\backslash e)| = |E(G)| - 1$, and if e is a link then $|V(G/e)| = |V(G)| - 1$. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by repeatedly contracting edges. We say that G has an H *minor* if G has a minor isomorphic to H.

Definition 1.2.2. Let G, H be graphs, and let $e \in E(H)$. We say that G is obtained from H by *subdividing* e if G is obtained from H by deleting e, adding a new vertex $w \notin V(H)$ and joining w to the ends of e by two new edges. Thus $|V(G)| = |V(H)|+1$ and $|E(G)| = |E(H)|+1$. The reverse operation, which (up to isomorphism) consists of contracting an edge incident with w, is called *supressing* w. We say that G is a *subdivision* of H if G is obtained from H by repeatedly applying the operation of subdividing and edge. We say that H *is topologically contained in* G, or that G *topologically contains* H or that G *has an* H *subdivision* if G has a subgraph isomorphic to a subdivision of H. Thus if G has an H subdivision, then it has an H minor.

Exercise 1.2.3. The statement "G has an H subdivision if and only if it has an H minor" holds for every G if and only if H has maximum degree at most three.

Exercise 1.2.4. A graph G has an H subdivision if and only if a graph isomorphic to H can be obtained from G by applying the following operations:
(i) deleting an edge,
(ii) supressing a vertex of degree two,
(iii) deleting an isolated vertex.

Exercise 1.2.5. A graph G has an H minor if and only if a graph isomorphic to H can be obtained from G by applying the following operations in arbitrary order:
(i) deleting an edge,
(ii) contracting an edge,
(iii) deleting an isolated vertex.

Exercise 1.2.6. A graph G has an H subdivision if and only if there is a 1–1 mapping $\phi : V(H) \to V(G)$ and for every $e \in E(H)$ there is a subgraph P_e of G such that
(i) if $e \in E(H)$ has distinct ends u, v, then P_e has ends $\phi(u), \phi(v)$,
(ii) if $e \in E(H)$ is a loop incident with $u \in V(H)$, then P_e is a cycle containing $\phi(u)$, regarded as a closed path,
(iii) for distinct $e, e' \in E(H)$ the paths $P_e, P_{e'}$ are internally disjoint.

Exercise 1.2.7. A graph G has an H minor if and only if for every $v \in V(H)$ there exists a connected subgraph G_v of G and a mapping $\psi : E(H) \to E(G)$ such that
(i) for distinct $u, v \in V(H)$, G_u and G_v are vertex-disjoint,
(ii) ψ is 1–1 and $\psi(e) \notin E(G_v)$ for every $e \in E(H)$ and $v \in V(H)$,

(iii) if $e \in E(H)$ has ends u, v, then $\psi(e)$ has one end in G_u and the other in G_v.

Exercise 1.2.8. Let L denote the graph with one vertex and one edge. The following conditions are equivalent.

(i) G has no L minor.
(ii) G has no L subdivision.
(iii) G is a forest.

Exercise 1.2.9. The following conditions are equivalent for every graph G:

(i) G has no K_3 minor,
(ii) G has no K_3 subdivision,
(iii) the underlying simple graph of G is a forest.

1.3. Characterization and recognition of series-parallel graphs

Definition 1.3.1. A graph is *series-parallel* if it has no K_4 subdivison.

Exercise 1.3.2. No series-parallel graph is 3-connected.

Definition 1.3.3. Let $k \geq 0$ be an integer, let G_1, G_2 be graphs on disjoint vertex-sets, and let A_i be a clique of size k in G_i. Let G be obtained from $G_1 \cup G_2$ by first identifying A_1 and A_2 into a set A, and then deleting an arbitrary subset of edges whose ends are distinct vertices of A. In those circumstances we say that G is a *k-sum* of G_1 and G_2. We also say that G is a *clique-sum* and a *$\leq t$-sum* of G_1 and G_2 for all $t \geq k$. Thus the result of this operation is not unique.

Exercise 1.3.4. Let $k \in \{0, 1, 2\}$, and let G be a k-sum of G_1 and G_2. Prove that if both G_1 and G_2 are series-parallel, then so is G. Prove that if G is series-parallel, and G_1, G_2 are k-connected, then both G_1 and G_2 are series-parallel.

Delete this exercise

Exercise 1.3.5. Let G be a 2-connected graph, let e be an edge of G with ends u, v, and assume that $G \backslash u \backslash v$ is disconnected. Prove that if $G \backslash e$ is series-parallel, then G is series-parallel.

Exercise 1.3.6. Prove that every simple series-parallel graph with at least two vertices has at least two vertices of degree at most two.

Definition 1.3.7. Let G be a graph. By a *series-parallel reduction* we mean any of the following operations:

(i) deletion of a loop,
(ii) deletion of a vertex of degree at most 1,
(iii) deletion of a parallel edge,
(iv) suppression a vertex of degree two.

We can now state the main characterization of series-parallel graphs, which will also explain where the name comes from.

Theorem 1.3.8. *A graph is series-parallel if and only if it can be reduced to the null graph by repeatedly applying the series-parallel reductions.*

Exercise 1.3.9. Prove Theorem 1.3.8.

Theorem 1.3.8 may be used to design a linear-time algorithm to test if an input graph is series-parallel. We leave this as an exercise, but it should be noted that the algorithm is less trivial than it may seem. How do you make sure that the total time spent on deleting parallel edges is at most linear?

Exercise 1.3.10. Find a linear-time algorithm to test whether an input graph is series-parallel.

We end this section with three other characterizations of series-parallel graphs stated in the form of exericses.

Exercise 1.3.11. A graph is series-parallel if and only if it can be obtained by repeated ≤ 2-sums starting from graphs on at most three vertices.

Exercise 1.3.12. A graph is series-parallel if and only if it can be obtained by repeated clique-sums starting from graphs on at most three vertices.

Exercise 1.3.13. A graph G is series-parallel if and only if it is a subgraph of a chordal graph with no K_4 subgraph. (A graph is *chordal* if it has no induced cycle of length four or more.)

Exercise 1.3.14. A 2-*terminal graph* is a graph G with two specified distinct vertices s and t, called the *source* and *sink*, respectively. A *series composition* of two 2-terminal graphs (G_1, s_1, t_1) and (G_2, s_2, t_2) is the 2-terminal graph with source s_1 and sink t_2 obtained from the disjoint union of G_1 and G_2 by identifying t_1 and s_2. The *parallel composition* of the same 2-terminal graph is obtained by identifying their sources and identifying their sinks to form the source and sink of the resulting 2-terminal graph. Prove that a 2-terminal graph G can be obtained from the unique 2-terminal graph (G, s, t) with two vertices and one edge by repeatedly applying the series and parallel compositions if and only if G is series-parallel and the graph $G + st$ has no cut vertex.

1.4. Edge-coloring series-parallel graphs

Many problems of interest are easy to solve using the results of the previous section, as indicated in the exercises at the end of this chapter. However, edge-coloring needs more work, and so we briefly discuss it. The edge-chromatic number of a graph G will be denoted by $\chi'(G)$.

Exercise 1.4.1. Prove that every non-null simple series-parallel graph has one of the following:
(i) a vertex of degree at most one,
(ii) two distinct vertices of degree two with the same neighbors,
(iii) two adjacent vertices of degree two,
(iv) a triangle with one vertex of degree two and one of degree three,
(v) five distinct vertices v_1, v_2, u_1, u_2, w such that the neighbors of w are u_1, u_2, v_1, v_2, and for $i = 1, 2$ the neighbors of v_i are w and u_i.

Exercise 1.4.2. Prove that every simple series-parallel graph G with no component isomorphic to an odd cycle satisfies $\chi'(G) = \Delta(G)$.

The situation is more complicated for graphs that are not simple.

Exercise 1.4.3. For $X \subseteq V(G)$ let $E(X)$ denote the set of edges with both ends in X. Thus $E(G) = E(X) \cup \delta(X) \cup E(V(G) - X)$. Prove that $\chi'(G) \geq |E(X)|/\lfloor |X|/2 \rfloor$ for every set $X \subseteq V(G)$ containing at least two elements. Prove that if $|X|$ is even, then this inequality is implied by the inequality $\chi'(G) \geq \Delta(G)$, and so gives no new information.

Remark 1.4.4. By the previous exercise, $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$, where

$$\Gamma(G) = \max\Big\{\frac{2|E(X)|}{|X|-1} : X \subseteq V(G), |X| \geq 3 \text{ and odd}\Big\}.$$

Exercise 1.4.5. Find a graph for which equality does not hold in the above.

Open Problem 1.4.6. Seymour conjectures that equality holds for every planar multigraph. (This implies the four color theorem.)

Open Problem 1.4.7. Seymour also conjectures that every multigraph G satisfies $\chi'(G) \geq \max\{\Delta(G)+1, \Gamma(G)\}$.

Exercise 1.4.8. Prove that Seymour's conjecture 1.4.6 holds for series-parallel graphs.

Definition 1.4.9. A graph G is k-edge-choosable, where k is an integer, if for every family of sets $(L_e : e \in E(G))$, each of size at least k, there exists a proper edge-coloring c such that $c(e) \in L_e$ for every edge $e \in E(G)$. Thus every k-edge-choosable graph is k-edge-colorable.

Open Problem 1.4.10. If a multigraph G is k-edge-colorable, then it is k-edge-choosable.

Exercise 1.4.11. Prove that 1.4.10 holds for simple-series-parallel graphs.

Open Problem 1.4.12. Does 1.4.10 hold for series-parallel multigraphs?

1.5. Exercises

Exercise 1.5.1. Given a graph G, two distinct vertices u, v of G, and a function $R : E(G) \to \mathbb{R}_0^+$, interpreted as the *resistance* of an edge, we can ask what is the total resistance between u, v in G. This resistance can be computed using Kirchhoff's laws, but sometimes one might be able to do it just by using the *series* and *parallel* rules. The former says that the total resistance of two edges in series of resistances r_1, r_2, is $r_1 + r_2$, the latter says that the total resistance of two edges in parallel of resistances r_1, r_2 is $1/(1/r_1 + 1/r_2)$. Prove that G is series-parallel if and only if for every two distinct adjacent vertices u, v of G and every assignment of resistances to the edges of G, the total resistance between u and v can be computed using the series and parallel rules.

Exercise 1.5.2. Prove that every loopless series-parallel graph has chromatic number at most three.

Exercise 1.5.3. Prove that every simple series-parallel graph on $n \geq 2$ vertices has at most $2n-3$ edges. Prove that the bound is best possible for every n.

Exercise 1.5.4. Given the algorithm from the previous exercise, one can solve many problems for series-parallel graphs in linear time. For instance:

(i) Independence number,
(ii) Vertex-cover (minimum number of vertices meeting all the edges),
(iii) Hamilton cycle (or path),
(iv) the presence of a subgraph isomorphic to a fixed graph H.

We will later extend this to graphs of bounded tree-width.

Definition 1.5.5. A graph G is *outer-planar* if it can be embedded in the plane (without crossings) with all its vertices on the boundary of the infinite face.

Exercise 1.5.6. Let G be a graph, and let H be obtained from G by adding a new vertex and joining it by an edge to every vertex of G. Then G is outer-planar if and only if H is planar.

Exercise 1.5.7. A graph G is outer-planar if and only if it has no K_4- or $K_{2,3}$ subdivision.

Hint. This can be proved directly or using Kuratowski's theorem.

Exercise 1.5.8. For every $n \geq 2$ construct a simple outer-planar graph on n vertices and $2n-3$ edges. This shows that the bound on the number of edges of a series-parallel graph cannot be improved even for outer-planar graphs.

Exercise 1.5.9. Find a linear-time algorithm to test outer-planarity.

1.6. Hints for selected exercises

Hint for Exercise 1.3.2. Take, if possible, a cycle C and a vertex $v \notin V(C)$, and study paths from v to C.

Hint for Exercise 1.3.4. If K is a K_4 subdivision in G, then one of $K \cap G_1$, $K \cap G_2$ is a subgraph of a path, and hence K can be converted to a K_4 subdivision in G_1 or G_2. For the converse, if J is a K_4 subdivision in G_1, say, then J is a subgraph of G unless $k = 2$ and J uses an edge e of G_1 that does not belong to G. But then e can be replaced by a path in G_2.

Hint for Exercise 1.3.5. If G has an H minor, say K, and $e \in E(K)$, then K is disjoint from some component of $G \backslash \{u, v\}$.

Hint for Exercise 1.3.6. Use Exercises 1.3.2 and 1.3.5 to break the graph into smaller graphs and apply induction.

Hint for Exercise 1.3.9. This follows immediately from Exercise 1.3.6.

Hint for Exercise 1.3.10. Allow the graph to have parallel edges. For every vertex $v \in V(G)$ introduce a variable $f(v)$, initially set to zero. **TO BE COMPLETED**

Hint for Exercise 1.4.1. See [**1**].

Hint for Exercise 1.4.5. If G is a 2-connected 3-regular graph, then $\Gamma(G) = 3$, so any 2-connected 3-regular graph that is not 3-edge-colorable is an example. In particular, the Petersen graph is an example.

Hint for Exercise 1.4.11. See [**1**].

Bibliography

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[2] V. G. Vizing, Critical graphs with given chromatic class, Metody Discret. Analiz. **5** (1965), 9–17.

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