1 Tree-width and grids

Scribe: Csaba Biró, February 28, 2005

Theorem 1.1 *The tree-width of the $k \times k$ grid is k.*

To show that k is an upper bound, we show a tree decomposition. Index the grid lines from 1 to k both horizontally and vertically; then (i, j) means the vertex in row i and in column j. The tree will be a path and the bags along the path are

$$\{(1,1),(1,2),\ldots,(1,k),(2,1)\},$$
$$\{(1,2),\ldots,(1,k),(2,1),(2,2)\},$$
$$\{(1,3),\ldots,(1,k),(2,1),(2,2),(2,3)\},\ldots,$$
$$\{(1,k),(2,1),\ldots,(2,k)\},$$
$$\{(2,1),\ldots,(2,k),(3,1)\},\ldots$$

For the lower bound, we introduce a game in which cops play against a robber. There are k cops and one robber. The game is played on a graph. At the first step, the robber picks a vertex and moves there. The cops will then announce all the vertices where they will move. The robber has the opportunity to move from his position; he can make arbitrarily many steps along edges. After this optional move, the cops will place their forces to the announced vertices. Then they announce their next move, which is lifting some units and replacing them to other vertices. They move in helicopters, so they don't have to move along edges. Then again, the robber has the opportunity to make several steps along edges, then the cops make the announced move, they announce their next move, and so on.

Every unit is visible to each player. There may be several cops in a vertex at one time.

The cops win if at any time, the robber will occupy a vertex where there is a cop. This also means that when the robber makes his several steps after the cops' announcement, he can't move through a vertex that is occupied by a cop. The robber wins if he can avoid being caught indefinitely.

Observe, that with given cop positions, say X, all that matters is in which component of $G \backslash X$ the robber is.

It is easy to see that on a tree with 2 cops, the cops have winning strategy. The next observation is an easy generalization.

Observation 1.2 If $\operatorname{tw}(G) \leq k$ then $k+1$ cops win.

Proof. Let (T, W) be a tree decomposition. Pick $t \in V(T)$ and land cops at W_t. Let $T_1, \ldots, T_l$ be the components of $T \backslash t$.

We claim that for every component K of $G \backslash W_t$ there exists $i \in \{1, 2, \ldots, l\}$ such that $V(K) \subseteq \bigcup_{t \in V(T_i)} W_t$. If not, then there exist $u, v \in V(K)$ such that $u \in \bigcup_{t \in V(T_j)} W_t$, $v \in \bigcup_{t \in V(T_{j'})} W_t$ for $j \neq j'$. Since K is connected, we may choose u, v adjacent. Let $r \in V(T)$ such that $u, v \in W_r$. We may assume $r \notin V(T_j)$. By the third axiom of tree decomposition, $u \in W_t$, a contradiction.

Let $t_2 \in V(T_i)$ be the neighbor of t in T_i. Let cops sitting in $W_t - W_{t_2}$ take off and land on $W_{t_2} - W_t$. The cops who stay on $W_t \cap W_{t_2}$ are enough to prevent the robber to "run away" (more precisely the component of the robber after the step will be a subset of K; proof is same as above). The game ends when cops reach a leaf; they capture the robber. $\square$

Now let's return to our original goal, that is to prove that the $k \times k$ grid has tree-width at least k. It suffices to describe an escape startegy for the robber against k cops on the grid.

We will show a strategy for $k-1$ cops. The startegy for k cops is similar, but more technical, so we leave it as an exercise.

Strategy: The number of rows is greater than the number of cops, and so there is at least one cop-free row, and similarly, there is at least one cop-free column. Go to the intersection of those, say position (i, j). When cops announce their move, find a cop-free row and column in the new position, say (i', j'). Then move horizontally to (i, j') and then vertically to (i', j').

Let us quote two corollaries without proof.

Corollary 1.3 *If a graph has a large grid minor, then it has a large tree-width.*

Theorem 1.4 (Robertson, Seymour) *There exists a function f such that every graph of tree-width at least $f(k)$ has a $k \times k$ grid minor.*

Comments: Current upper best bound is $f(k) = 20^{2k^5}$. Open question whether there is a polynomial bound. A lower bound is $\Omega(k^2 \log k)$.

We will use the following theorem and lemma to deduce another corollary. We do not provide a proof for the theorem.

Theorem 1.5 (Tutte) *Every 4-connected planar graph is Hamiltonian.*

Lemma 1.6 *For every planar graph G there exists k such that G is isomorphic to a minor of a $k \times k$ grid.*

Proof. We may assume G is a simple triangulation.

We claim that there exists a 4-connected planar triangulation G' such that $G \leq_m G'$. To see this, first observe that the only possible violations of 4-connectivity in G are separating triangles. If not, suppose there is a 3-vertex-cut. Pick two arbitrary vertices of the cut u, v. Since the graph is planar, there is simple closed curve between u and v which goes through only one face. But that face is also a triangle, which shows that $u \sim v$. Thus in fact 3 vertices of the cut form a separating triangle.

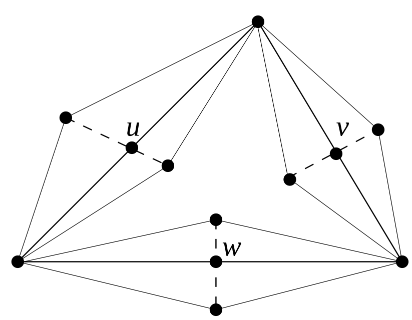

Figure 1: Eliminating separating triagles

Now consider a separating triangle. The construction on Fig. 1 shows how to eliminate it, forming a new graph that contains the original as a minor. (The new vertices are u, v and w, the new edges are the broken lines) This way we can eliminate separating triangles one at a time until we reach a 4-connected triangulation.

By Tutte's theorem it is enough to show the claim for Hamiltonian graphs. A Hamiltonian triangulation consists of the Hamiltonian cycle and some chords inside and some chords outside. Take a grid which is fine enough, assign the Hamiltonain cycle to the half of the grid and then draw the chords on the grid accordingly. See Fig. 2.

Corollary 1.7 *For every planar graph H there exists and integer M such that every G of tree-width at least M has a minor isomorphic to H.*

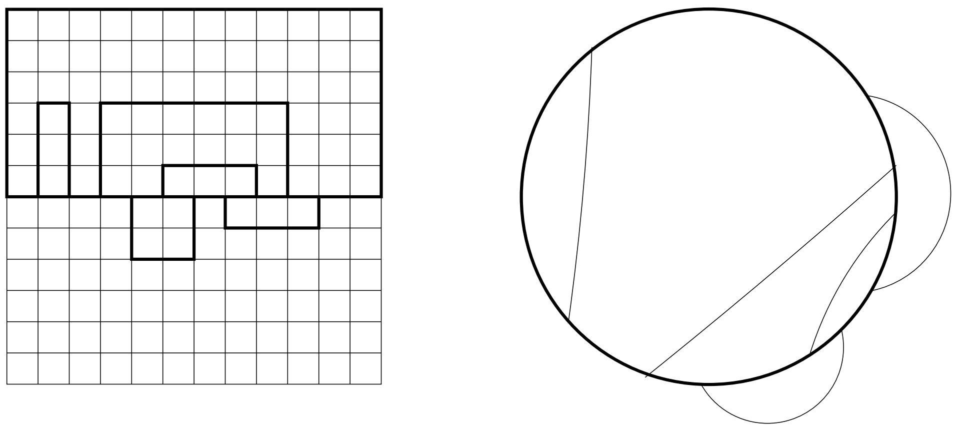

Figure 2: Finding the grid for Hamiltonian cycles

Remark: For non-planar H no such integers exist. (Think grids.)

Theorem 1.8 (Robertson, Seymour) *For every planar graph H there exists a function g such that for every graph G and every integer k, either G has k disjoint H minors, or it has a set X of at most $g(k)$ vertices such that $G \backslash X$ has no H minor.*

Remark (1): This is a generalization of the Erdős–Pósa Theorem, which is the above when H is a loop.

Remark (2): False for every non-planar graphs H. E. g. to see this for $H = K_5$, let G be a very fine grid on the torus. It has no two disjoint K_5 minors, but $|X|$ can be arbitrarily large, so $g(2)$ does not exist.

Proof. Let M be such that every graph G of tree-width at least M has a minor isomorphic to k disjoint copies of H. We may assume $\mathrm{tw}(G) < M$, for otherwise we are done.

Let (T, W) be a tree decomposition of G of width less than M. For every H minor of G, say J, let T_J be the set of all vertices $t \in V(T)$ such that $W_t \cap V(J) \neq \emptyset$. We may assume that H is connected and then J is connected, hence T_J is the vertex set of a subtree of T. By an easy extension of a standard graph theory exercise, either there are $J_1, J_2, \ldots, J_k$ such that $T_{J_1}, T_{J_2}, \ldots, T_{J_k}$ are disjoint or there is a set of k vertices of T intersecting all $\{T_J\}_J$. In the former case $J_1, J_2, \ldots, J_k$ are disjoint H minors of G; in the latter, let X be the mentioned subset of T. Then

$$\bigcup_{t \in X} W_t$$

is a set of at most kM vertices of G that intersects all H minors.