Math 8863 Topics in Graph Theory **Spring 2007**

Lecture 1: January 31, 2007

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1.1 Introduction

The proof of Kuratowski's theorem provides us with a template for an algorithm that tests whether a graph is planar. There is an algorithm of Hopcroft and Tarjan that checks 3-connectivity in linear time. Note that a quadratic algorithm is easy because it is possible to check 2-connectivity in linear time. In this lecture we will describe a quadratic-time algorithm (with respect to the number of vertices in the graph) that tests whether a graph has a planar drawing. We note that there exist linear-time planarity algorithms, but they are much more complicated than the quadratic one described here.

1.1.1 Background Definitions

Definition 1.1 *Let $H \subseteq G$. An* H-bridge *of G is a subgraph B of G such that either:*
(i) $B \cong K_2$ with both vertices but not its edge in H, or
(ii) B consists of a component of $G \setminus V(H)$, all edges from that component to H and all ends of those edges. The vertices of $V(B) \cap V(H)$ are called vertices of attachment *or simply* attachments *of B. (Figure 1)*

Notice that bridges can only share vertices of attachment with other bridges, and furthermore, bridges are edge-disjoint from H. In addition, bridges play an important role in growing planar subgraphs. One characteristic used to describe the process of adding bridges to a graph is that of f-admissibility.

Definition 1.2 *Let H be a plane graph and let $H \subseteq G$. For $f \in F(H)$, we say that an H-bridge B is* f-admissible *if all the attachments of B belong to the boundary of f.*

1.2 The Algorithm

To test planarity, it suffices to test each block separately, and it suffices to restrict ourself to 2-connected graphs. Consider the following algorithm, which given a 2-connected graph, returns either a planar drawing of G or a statement that no such drawing exists:

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PLANAR($G$)

1. Let $H$ be a cycle in $G$ (this exists since $G$ is a 2-connected graph).  Pick a planar drawing
   of $H$.

2. If there is no $H$-bridge, i.e.  $H = G$, then $H$ is a planar drawing of $G$.
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3. If some H-bridge is f-admissible for no face f, then output NO and stop.
4. If some H-bridge is f-admissible for a unique face $f \in F(H)$, then let B be one such bridge. Otherwise let B be any H-bridge.
5. ~~Pick a nontrivial sub-path P of B with ends in H and otherwise disjoint from H.~~
6. Let $H := H \cup P$. The sub-path P may not be unique and we make an arbitrary choice if there is one. Draw P in any face f so that B is f-admissible.
7. Return to step 2.

1.3 Proof of Correctness

We will show that if G is planar, then for every H in the algorithm, the planar drawing of H can be extended to some planar drawing of G. (Recall that the same graph can have different drawings.)

To begin the proof, suppose that the planar drawing of H can be extended to some planar drawing of G. We must show that this property holds for $H \cup P$, where B, f and P are given in the description of the algorithm. Let Γ be a drawing of G that extends H. In Γ, the bridge B lies in a face f' of H. If $f = f'$, then Γ gives a planar drawing that extends $H \cup P$.

Therefore, we may assume that $f \neq f'$. Also note that the bridge B is both f and f'-admissible. The algorithm drew the path P in the face f, so B is f-admissible, and in the drawing, Γ, the bridge B is drawn in the face f', so B is f'-admissible. This implies that for every bridge B', there exist at least two faces, g, such that B' is g-admissible. Otherwise, we would have been forced by the algorithm to select the bridge that was admissible in only one face and $f = f'$.

Let Γ' be obtained from Γ by swapping all H-bridges that are both f and f' admissible. Specifically, if an H-bridge that is both f and f'-admissible is originally drawn in f, it is now drawn in f' and vice versa. There are two situations that could cause a problem. The first problem arises when the swapped bridge could not attach the same way in f and f'. However, this problem can not occur because the attachments of a bridge always come in the same cyclic order in f and f'. This consistent ordering can be seen when the bridge is "flipped" from face f to f'. (Figure 2)

The second problem occurs when we could produce an obstruction. Specifically, the bridge that is swapped from f to f' (or f' to f) may be forced to have edges cross with a bridge only admissible in exactly one of f, f'. To rectify this problem, consider the following two mutually exclusive definitions that describe how two bridges can produce a crossing.

Definition 1.3 *Two bridges B and B' are* skew *if they have the same set of attachments and each has exactly three attachments. (Figure 3)*

Definition 1.4 *Two bridges B and B'* overlap *on a face f if there exist distinct vertices a, b, c, d on the boundary of f in order, such that a and c are attachments of B and b and d are attachments to B'. (Figure 4)*

If we have an obstruction in the drawing Γ', then there are two bridges D_1 and D_2 that are skew or overlap. Without loss of generality, suppose that D_1 is the bridge that did not swap faces and let D_2 be the newly inserted bridge. Also, without loss of generality suppose that in Γ', D_1 and D_2 are drawn in f. So D_2 is f and f'-admissible, but D_1 is only f-admissible. By the construction of the algorithm, we know that besides f, D_1 is admissible in another face, call it $f'' \neq f$. Note that $f'' \neq f'$ as D_1 is not f' admissible.

The bridges D_1 and D_2 cannot be skew because one is f' admissible and the other is not. As a result, D_1 and D_2 can not share the same attachment set. Thus, they overlap on the face f. In addition, define a and c to be vertices of attachment of D_1 to face f and define b and d to be vertices of attachment of D_2 to face f. Furthermore, we name these vertices so that their order around face f is a, b, c, d. Notice that vertices a and c are also contained in f'' and vertices b and d are contained in f'. Let v be a vertex in f and draw edges from v to a, b, c and d. Then add an edge drawn inside f' between vertices b and d and an edge drawn inside f'' between vertices a and c. (These edges are drawn as dashed lines in Figure 5.) This produces a subdivision of K_5 in the plane.

As a result, we have shown that Γ' is a planar drawing and it extends $H \cup P$, completing the proof.

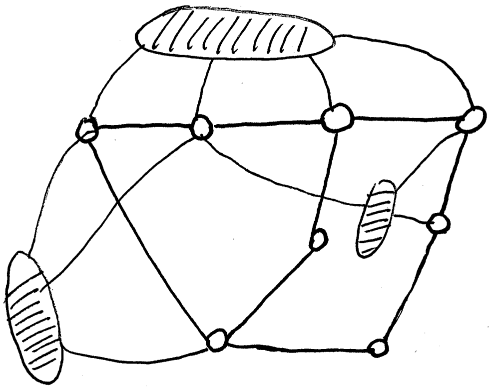

Figure 1

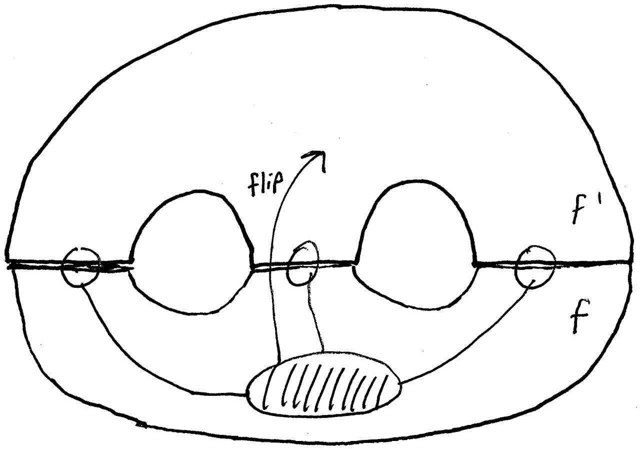

Figure 2

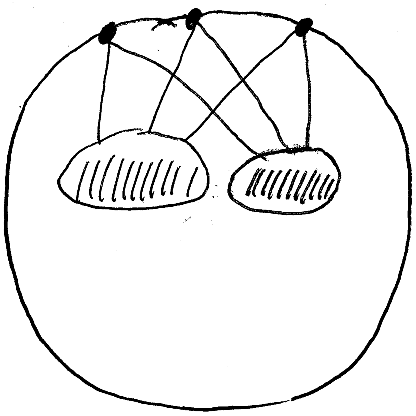

Figure 3

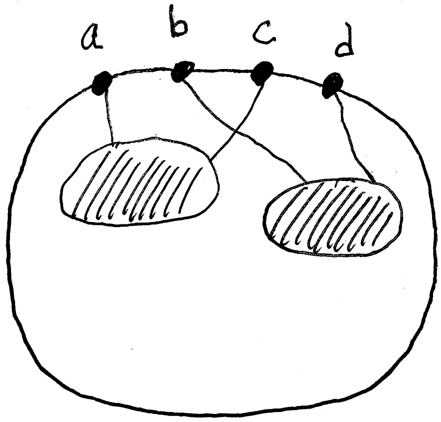

Figure 4

Figure 5