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# Planar Graphs and Planar Posets - I

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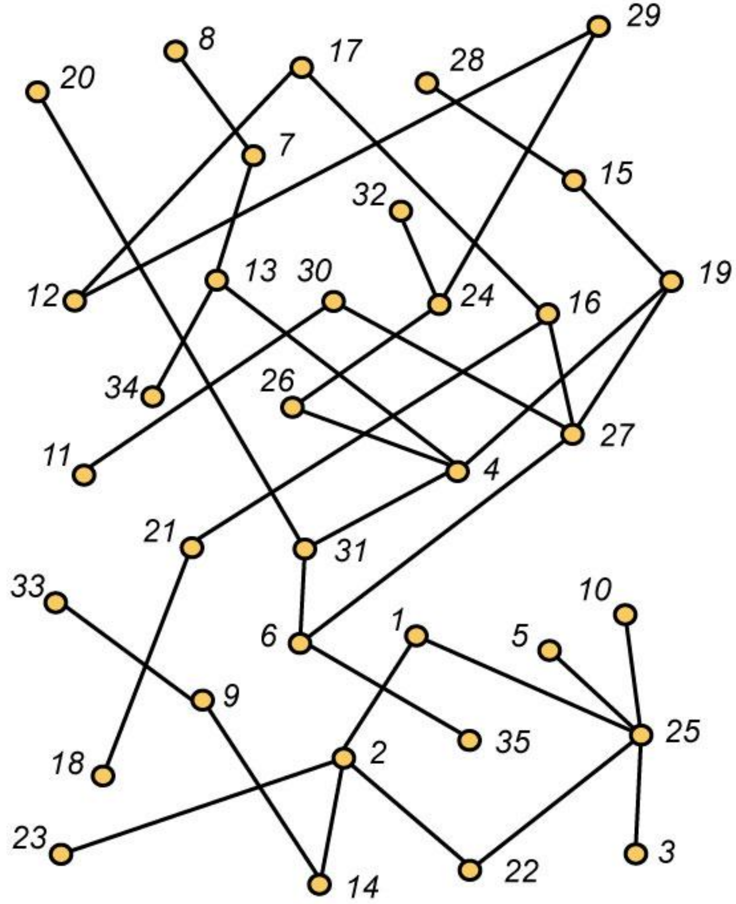
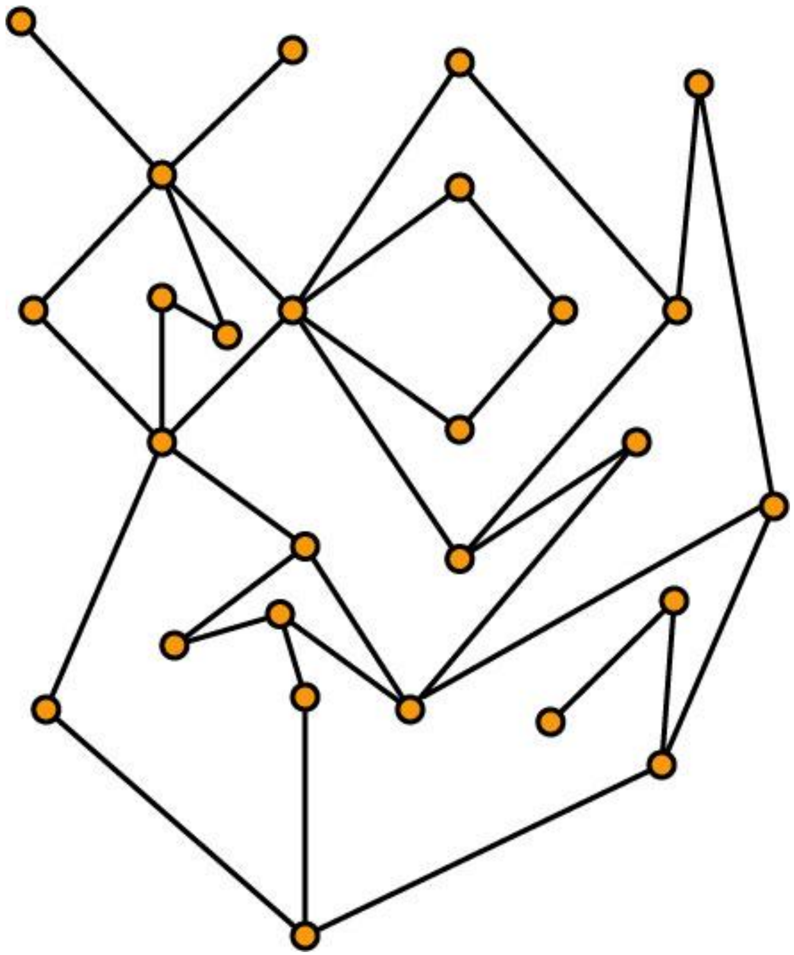
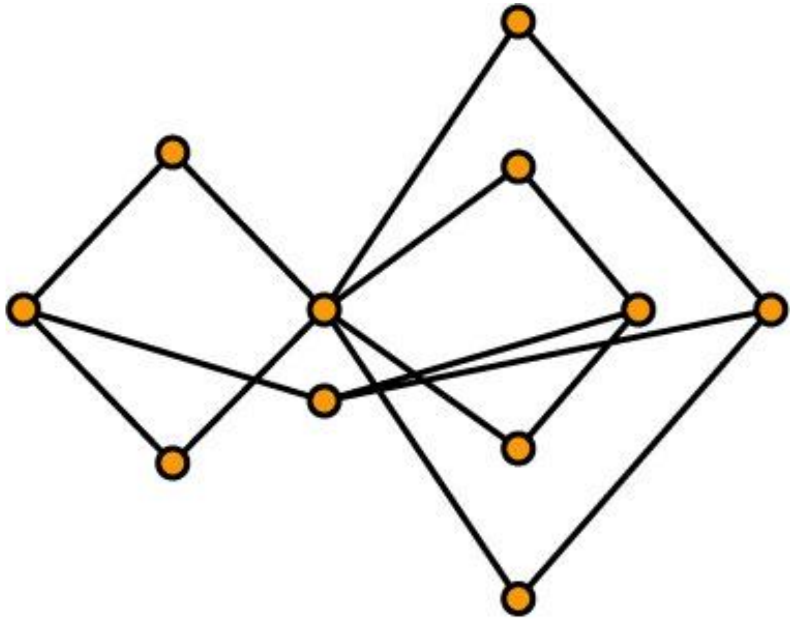
# Partially Ordered Sets



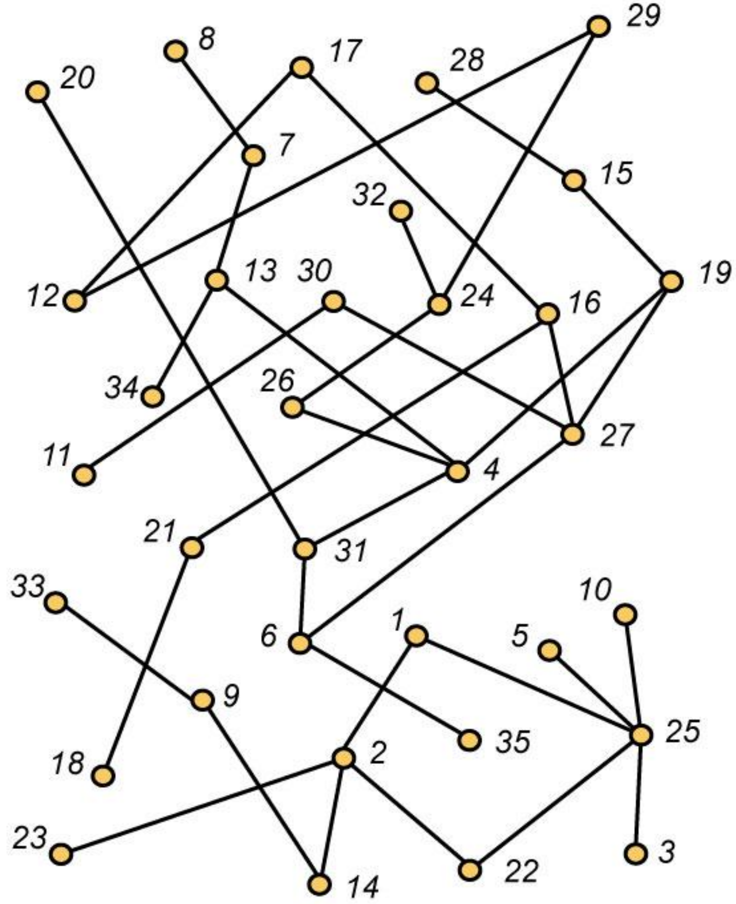
# A Planar Poset



# A Non-planar Poset



# This Poset is Planar!



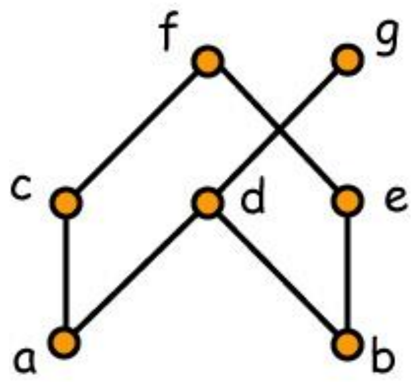
# Complexity Issues

**Fact** (Garg and Tamassia) The question "Does $P$ have a planar Hasse diagram?" is NP-complete.

**Theorem** (Brightwell) The question "Is $G$ a cover graph?" is NP-complete.

# Realizers of Posets

A family **F** $= \{L_1, L_2, \ldots, L_t\}$ of linear extensions of $P$ is a realizer of $P$ if $P = \cap$ **F**, i.e., whenever $x$ is incomparable to $y$ in $P$, there is some $L_i$ in **F** with $x > y$ in $L_i$.




$L_1 = b < e < a < d < g < c < f$

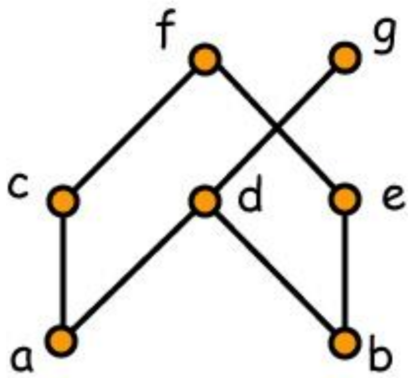
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

$L_4 = b < e < a < c < f < d < g$

$L_5 = a < b < d < g < e < c < f$

# The Dimension of a Poset




$L_1 = b < e < a < d < g < c < f$

$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

The dimension of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \le 3$. In fact,

$$\dim(P) = 3$$

# Complexity Issues

**Theorem** (Yannakakis) For fixed $t \geq 3$, the question $\dim(P) \leq t$ ? is NP-complete.

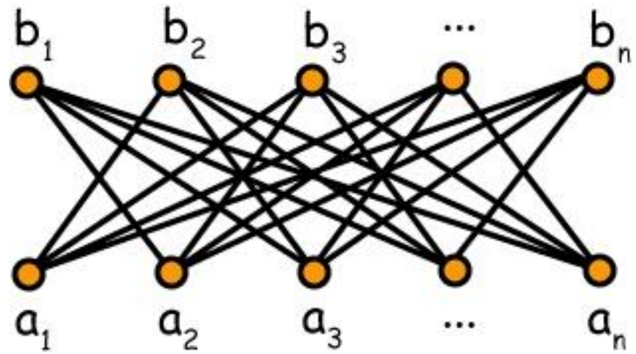
**Theorem** (Yannakakis) For fixed $t \geq 4$, the question $\dim(P) \leq t$ ? is NP-complete, even when $P$ has height 2.

# Testing $\dim(P) \leq 2$

**Fact** Testing a graph on $n$ vertices to determine whether it is a comparability graph can be done in $O(n^4)$ time.

**Fact** A poset $P$ satisfies $\dim(P) \leq 2$ if and only if its incomparability graph is a comparability graph.

# Standard Examples



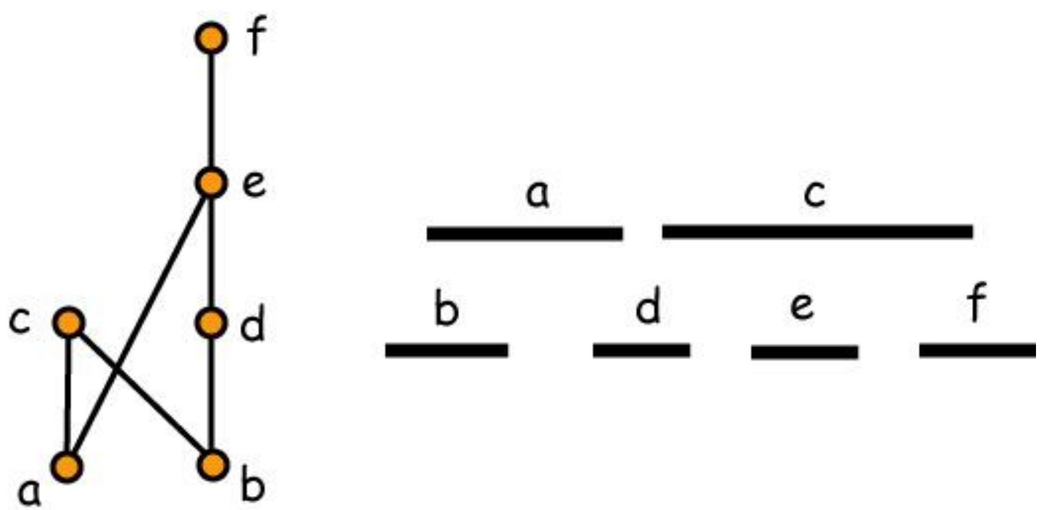

$S_n$

**Fact** For $n \geq 2$, the standard example $S_n$ is a poset of dimension $n$.

**Note** If $L$ is a linear extension of $S_n$, there can only be one value of $i$ for which $a_i > b_i$ in $L$.

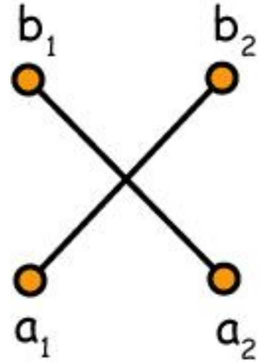
# Interval Orders

A poset $P$ is an interval order if there exists a function $I$ assigning to each $x$ in $P$ a closed interval $I(x) = [a_x, b_x]$ of the real line $\mathbf{R}$ so that $x < y$ in $P$ if and only if $b_x < a_y$ in $\mathbf{R}$.



# Characterizing Interval Orders

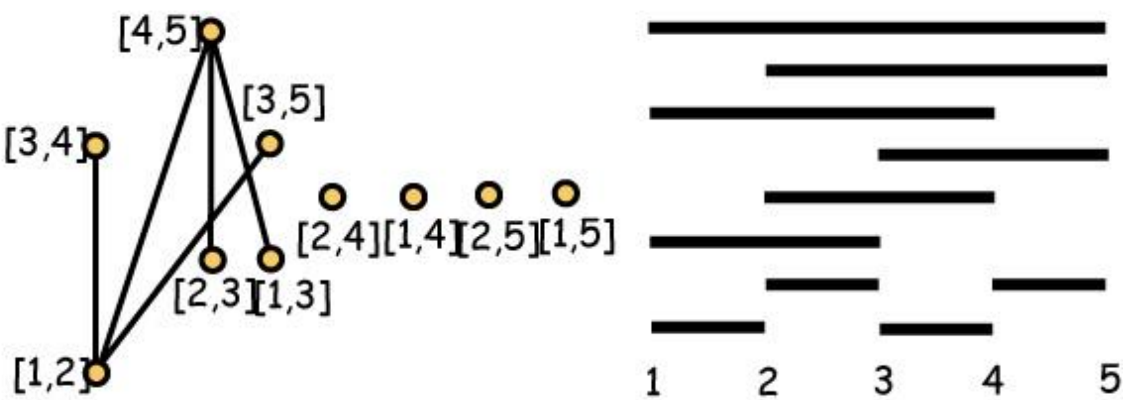
**Theorem** (Fishburn) *A poset is an interval order if and only if it does not contain the standard example* $S_2$.




$$S_2 = 2 + 2$$

# Canonical Interval Orders

The canonical interval order $I_n$ consists of all intervals with integer end points from {1, 2, …, n}.




$I_5$

# Dimension of Interval Orders

**Theorem** (Füredi, Rödl, Hajnal and Trotter) The dimension of the canonical interval order $I_n$ is

$$\lg \lg n + (1/2 + o(1)) \lg \lg \lg n$$

# Dimension and Height

**Observation** Posets of height 2 can have arbitrarily large dimension … but among the interval orders, large dimension requires large height.

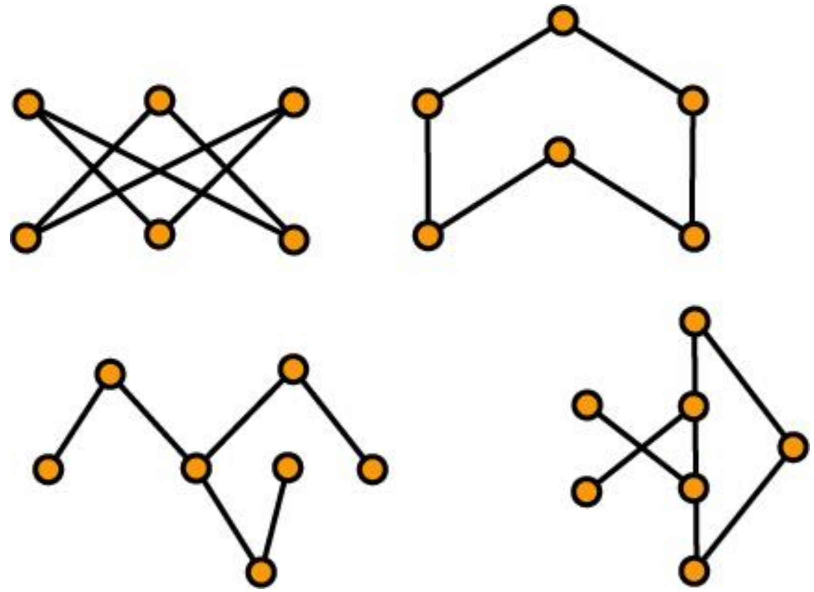
**Fact** For every $g$, $t$, there exists a poset $P$ of height 2 for which $\dim(P) > t$ and the girth of the comparability graph of $P$ is at least $g$.

# Adjacency Posets

The adjacency poset $P$ of a graph $G = (V, E)$ is a height 2 poset with minimal elements $\{x' : x \in V\}$, maximal elements $\{x'' : x \in V\}$, and ordering: $x' < y''$ if and only if $xy \in E$.

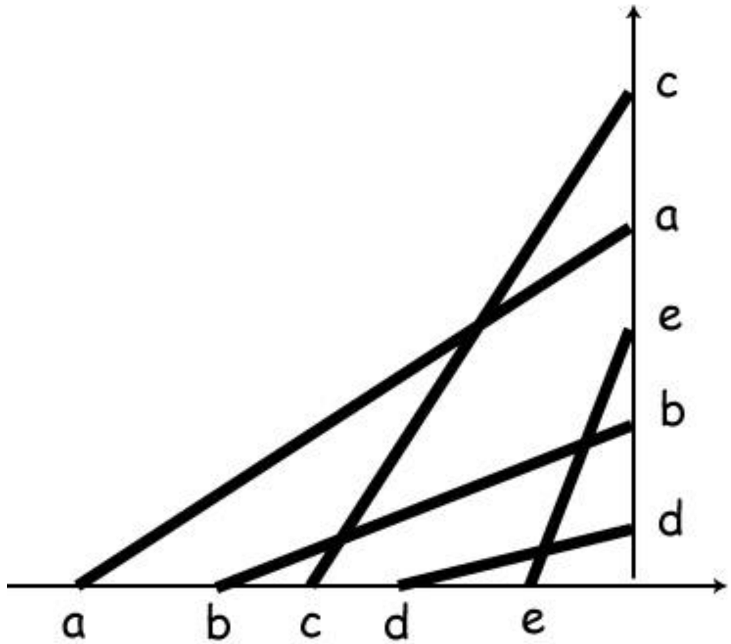
**Fact** If $P$ is the adjacency poset of a graph $G$, then $\dim(P) \geq \chi(G)$.

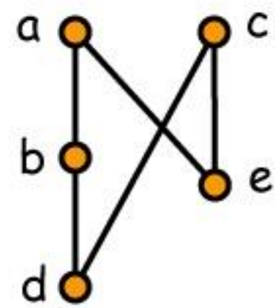
# Irreducible Posets



**Fact** These posets are irreducible and have dimension 3. The full list of all such posets is known. It consists (up to duality) of 7 infinite families and 10 other examples.

# Posets of Dimension at most 2

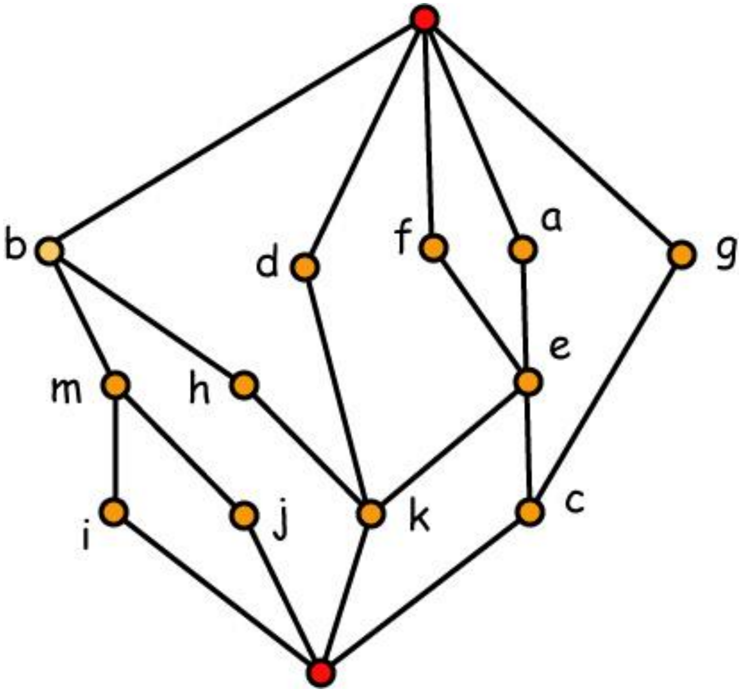


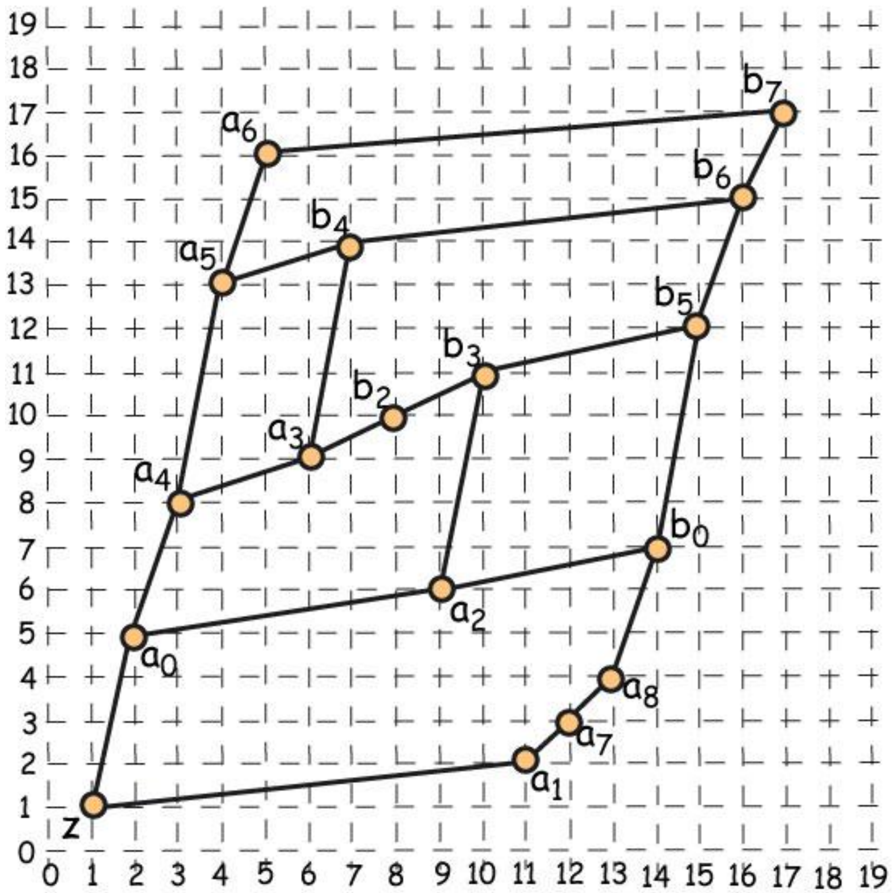

**Fact** A poset P has such a representation if and only if it has dimension at most 2.

# Dimension of Planar Posets

**Theorem** If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.

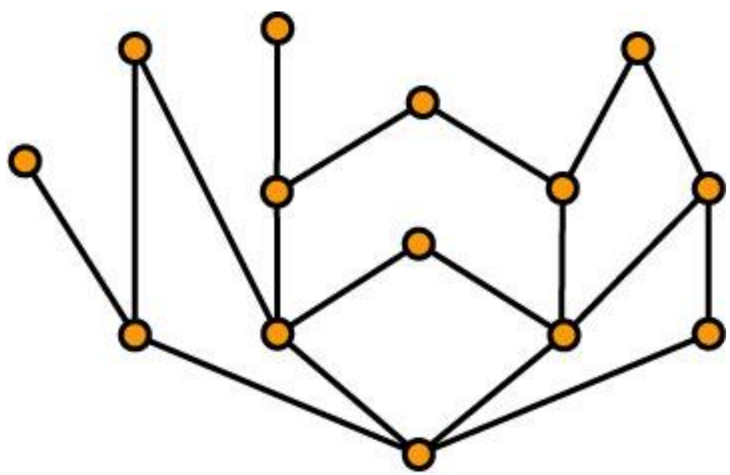


# Dimension of Planar Posets (2)



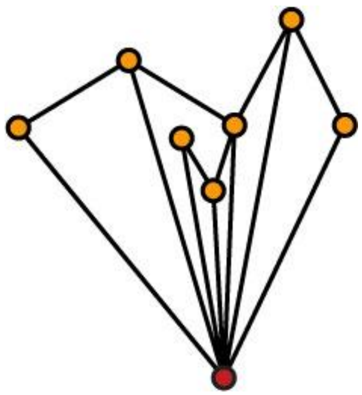
# Dimension of Planar Posets (3)

**Theorem** (Trotter and Moore) If $P$ has a $0$ and the diagram of $P$ is planar, then $\dim(P) \leq 3$.



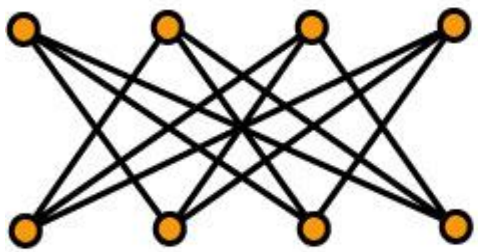
# The Dimension of a Tree

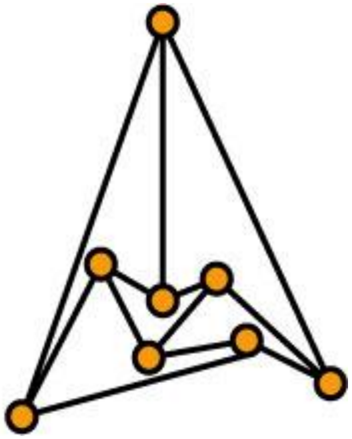
**Corollary** (Trotter and Moore) If the diagram of $P$ is a tree, then $\dim(P) \leq 3$.



# A 4-dimensional planar poset

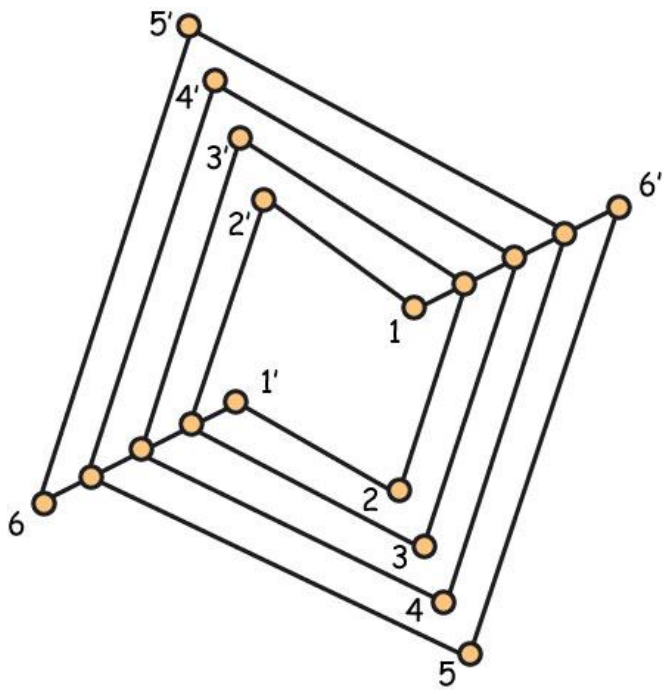
**Fact** The standard example $S_4$ is planar!.





# Kelly's Construction

**Fact** For every $t$, the standard example $S_t$ is contained in a planar poset.



# Questions

1. Under what conditions does a poset with a planar cover graph also have a planar diagram?
2. Which posets are subposets of planar posets?
3. For each $t \geq 4$, what is the smallest planar poset having dimension $t$?
4. Does a planar poset of large dimension necessarily have large height?