Yet another talk on the on-line chain partitioning of posets

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Outline

Chain partitioning game

- Two person game.
- The number of chains c is a parameter of the game.
- Spoiler reveals one point at a time of a poset.
- Algorithm puts it into a chain $1, \ldots, c$.
- If Algorithm can't make a move, he loses.
- If they play forever, Spoiler loses.

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If Spoiler is not allowed to build an anitchain of size more than w, how many chains does Algorithm need to play the game forever?

Known bounds

Theorem (Kierstead, 1981)

For every $w \geq 1$ there exists an algorithm which constructs an on-line partition of a poset of width w into $\frac{5^w-1}{4}$ chains.

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There is an algorithm and a c constant such that the algorithm constructs $w^{c \log w}$ chains.

Theorem (Szemerédi)

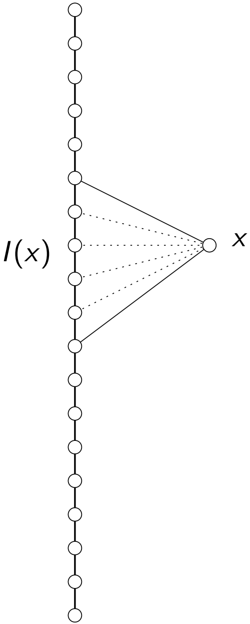
For every $w \geq 1$ and every on-line algorithm $\mathcal{A}$ there exists an algorithm that builds a poset of width at most w, but $\mathcal{A}$ uses at least $\frac{w(w+1)}{2}$ chains.

Sketch of Kierstead's proof

Base case: $w = 2$.

- $$\frac{5^2 - 1}{4} = 6$$
- We start building a "greedy chain" C_1. If a new point is comparable to every element of C_1, it goes into C_1.
- The rest will go into one of C_2, C_3, C_4, C_5, C_6 according to the following strategy.

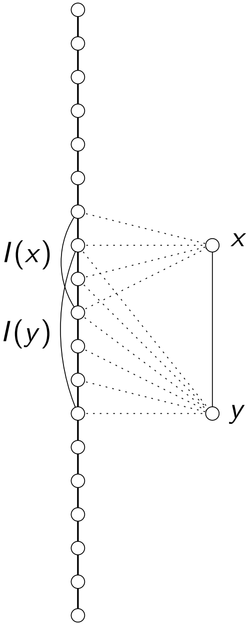
Fact 1



Proposition

If x can't go into C_1, then $I(x) = \{z \in C_1 : z \| x\}$ is an interval of C_1.

Fact 2

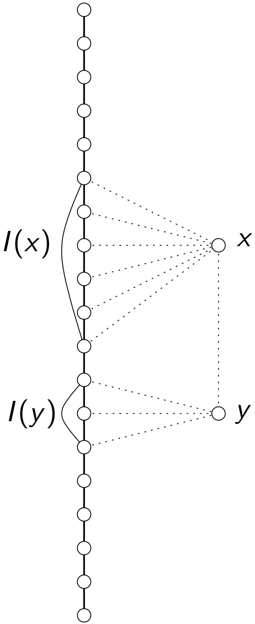


Proposition

If $y < x$ for $x, y \notin C_1$, then

- *The lowest point of $I(y)$ is below (or same point as) the lowest point of $I(x)$.*
- *The highest point of $I(y)$ is below (or same point as) the highest point of $I(x)$.*
- *They may or may not intersect.*

Fact 3



Proposition

If $y \| x$ *for* $x, y \notin C_1$*, then* $I(x) \cap I(y) = \emptyset$.

The $*$-order

Definition

For $P - C_1$ we define $x * y$ if

1) $x < y$ in P *or*
2) $I(x) < I(y)$.

Proposition

$(P - C_1, *)$ *is a total order.*

Classes of $P - C_1$

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- If x and y are consecutive elements of the same class, then $I(x) \cap I(y) \neq \emptyset$.

Remark: the second property implies that consecutive elements are comparable in P, so every class is a chain in P.

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Definition (of the classes)

- When the new element x comes in into the middle of class A, we put it into class A.
- When x comes in between classes, and $x <: y$ in $*$, and $I(x) \cap I(y) \neq \emptyset$, then we put x into the class of y.
- If no such y exists, but $z <: x$ in $*$, and $I(z) \cap I(x) \neq \emptyset$, then we put x into the class of z.
- If no such z exist, then we start a new class for x.

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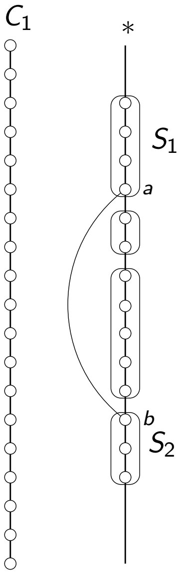
A proof is necessary to show that the two properties are maintained.

Far classes are comparable



Proposition

If S_1 and S_2 are classes with at least two other classes between them, then $S_1 \cup S_2$ is a chain.

Strategy for $w = 2$

- New element x comes in.
- If we can put x into C_1, we will.
- If not, we compute $*$, and we find the class of x.
- If x is joining an existing class A, we put it into the chain of A. (So the property, that every class uses only one chain is maintained.)
- If x starts its own class X, and there are at most 4 other classes, we put x into a new chain.
- If there are at least 5 other classes, then we identify the chain indices of the "close" classes (at most 4), and use a different one.

The new $*$-order

Start building a greedy chain C_1, and define the $*$-order on $P - C_1$:

Definition

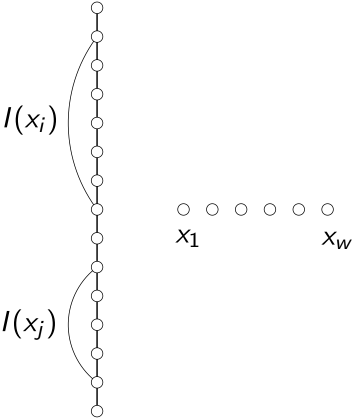
We say $x * y$ if

1) $x < y$ in P or
2) $I(x) < I(y)$ or
3) $\exists u \in P - C_1 : x < u$ in P and $u * y$ or
4) $\exists v \in P - C_1 : x * v$ and $v < y$ in P.

Remark

- It may happen that $x \| y$, but $I(x) \cap I(y) \neq \emptyset$.
- Therefore $*$-order is not a chain, furthermore 1) and 2) is not enough for transitivity.
- Nevertheless, $(P - C_1, *)$ is a poset.

Width of the $*$-order



Proposition

$(P - C_1, *)$ is of width at most $w - 1$.

Final steps

Hence it is possible to partition $P - C_1$ into $\frac{5^{w-1}-1}{4}$ chains in the $*$-order.

Proposition

Far classes on each $*$-chain are comparable.

Repeat the construction to partition each $*$-chain into 5 chains in P.

$$5 \cdot \frac{5^{w-1} - 1}{4} + 1 = \frac{5^{w} - 1}{4}$$

Q.E.D.

Reference poset

- Pick a reference poset A with $w(A) = w - 1$ greedily.
- Suppose we can partition the rest into $p(w)$ chains.
- $f(w) = f(w - 1) + p(w)$.

Conclusion: if we believe that there is a polynomial algorithm, then we still get a polynomial algorithm after the greedy step.

Let $R_x = \{z \in A : z \| x\}$.

Lemma

Let $x, y \in P \setminus A$.

1. $w(R_x) = w - 1$.
2. *If $x \| y$ then $w(R_x \cap R_y) \leq w - 2$.*
3. *Let C be a chain in A. Then $R_x \cap C$ is an interval of C.*
4. *If $x < y$ and C is a chain such that $x, y \notin C$, then* $\max\{R_x \cap C\} \leq \max\{R_y \cap C\}$ *and* $\min\{R_x \cap C\} \leq \min\{R_y \cap C\}$.

Lemma

Let $x, y \in P \setminus A$, $x \| y$, let $\{x_1, x_2, \ldots, x_{w-1}\}$ be an antichain in R_x and $\{y_1, y_2, \ldots, y_{w-1}\}$ be an antichain in R_y. The following statements are equivalent.

i) *There exists an index $i_0 \in [w-1]$ satisfying $x_{i_0} < y$.*

ii) *For any $i \in [w-1]$, either $x_i < y$ or $x_i \| y$.*

iii) *There exists an index $j_0 \in [w-1]$ satisfying $y_{i_0} > x$.*

iv) *For any $j \in [w-1]$, either $y_j > x$ or $y_j \| x$.*

Definition

Let $x, y \in P \setminus A$, $x \| y$. Define $x\sigma y$ if there is an antichain $\{x'_1, x'_2, \ldots, x'_{w-1}\} \in R_x$ and an antichain $\{y'_1, y'_2, \ldots, y'_{w-1}\} \in R_y$ satisfying $x'_i \leq y'_i$ for all $1 \leq i \leq w - 1$.

Theorem

$$x\sigma y \Leftrightarrow \exists \textit{ antichains like in lemma}$$

Classes?

These are more like just ideas:

- Create classes by "representative antichains".
- It may be OK for a point to belong to several classes (it won't belong to more than two anyway).
- It is easy to prove this way that $f(2) \leq 6$.

Thank you!