News about Semiantichains and Unichain Coverings

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joint work with

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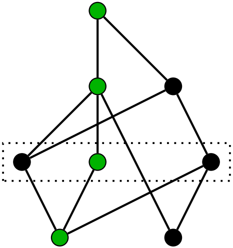
Basic properties – chains and antichains

a chain – a set where each two points are comparable
an antichain – a set where each two different points are incomparable

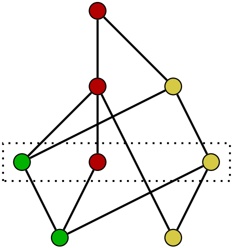


height – the size of the longest chain
width – the size of the longest antichain

Classical min-max theorem

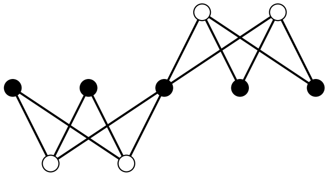
Theorem (Dilworth, 1950)

The size of maximum antichain is equal to the size of minimum chain covering.



Generalization of Dilworth's theorem

k-antichain – a set of k disjoint antichains
k-chain – a set of k disjoint chains



Theorem (Greene & Kleitman, 1976)

For every k there is a chain partition $\mathcal{C}$ such that the size of maximum k-antichain is equal to $\sum_{C \in \mathcal{C}} \min(k, |C|)$.

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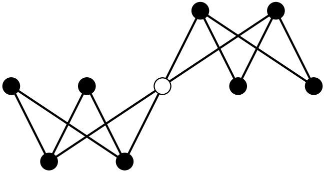

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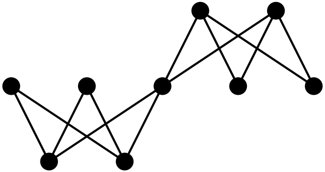

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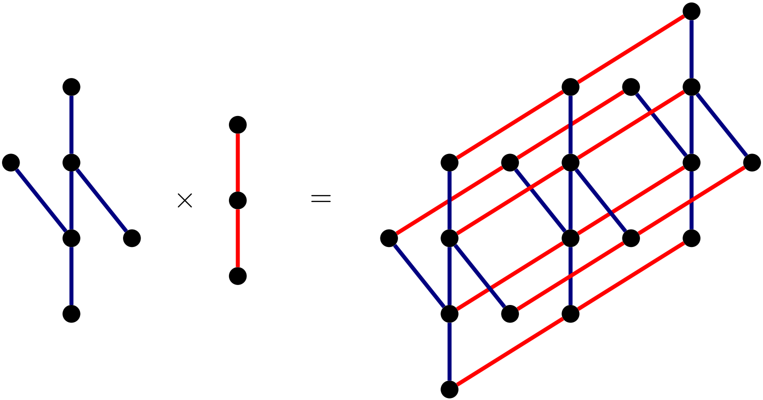
Cartesian product and the proof of Saks



Theorem (Saks, 1979)

In a product $C \times Q$ where C is a chain the size of maximum antichain equals the size of chain covering with chains of the form $C \times \{q\}$ and $\{c\} \times C'$ (called unichains).

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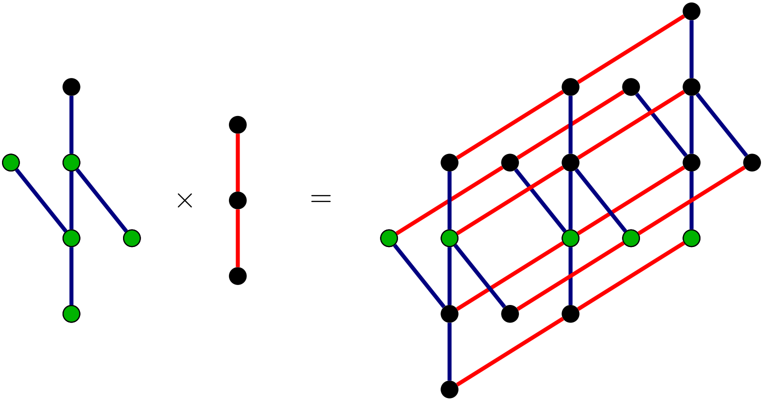

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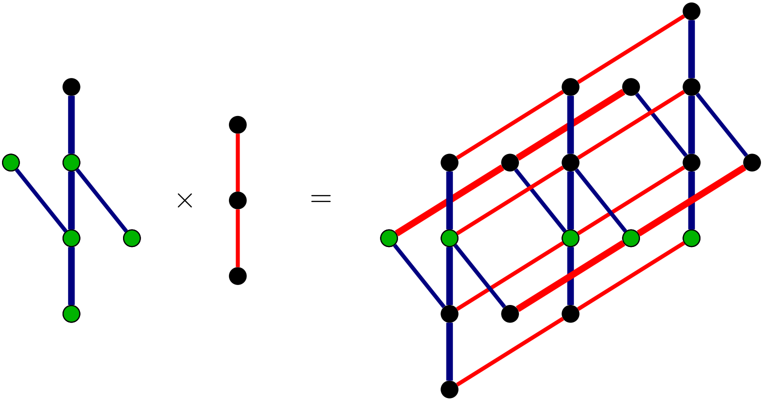

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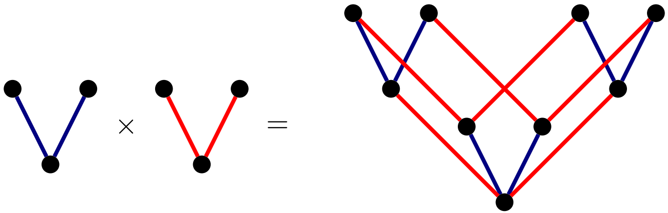
Semiantichain conjecture



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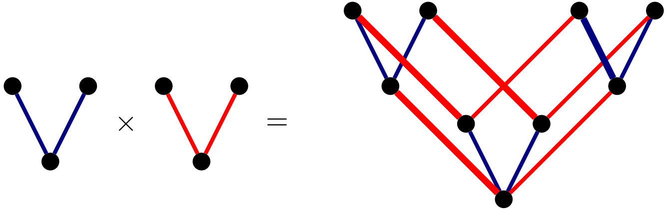

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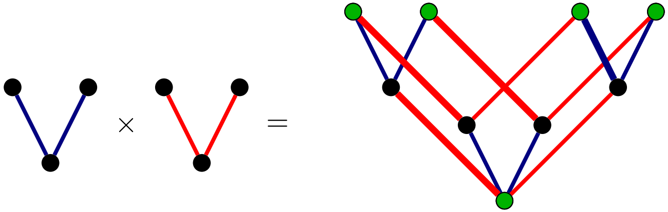
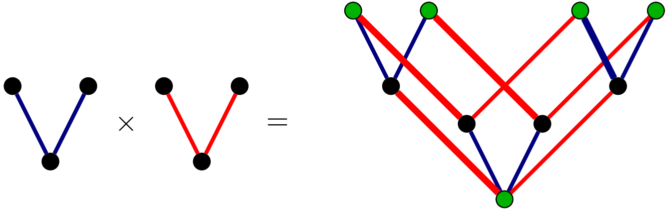


semiantichain – a set in which no two points are in a common unichain

Semiantichain conjecture



semiantichain – a set in which no two points are in a common unichain

Conjecture (Saks & West, 1980)

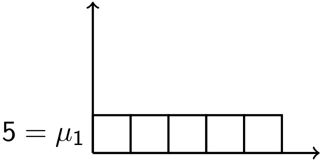
In the product $P \times Q$ the size of maximum semiantichain equals the size of minimum unichain covering.

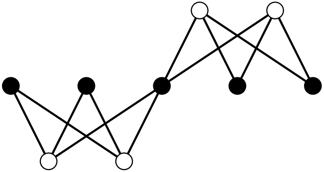
Old results

- Saks, 1979
  product $C \times Q$ of a chain C and an arbitrary poset Q
- Liu & West, 2008
  product of two posets of width $\leqslant 2$
- Liu & West, 2008
  product of two posets of height $\leqslant 2$
- West & Tovey, 1981
  other classes with more complicated properties

Greene diagram

$$\sum_{i=1}^{k} \mu_i := a_k = |\text{max. } k\text{-ant.}| \qquad \sum_{i=1}^{j} \lambda_i := c_j = |\text{max. } j\text{-ch.}|$$





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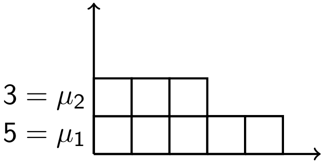

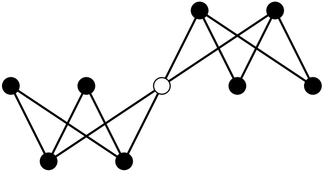

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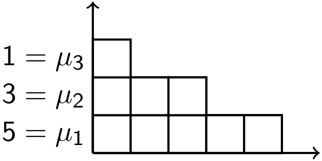

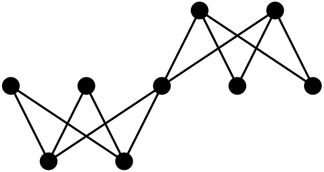

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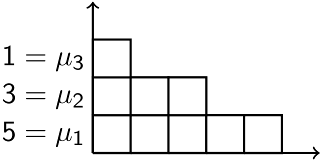

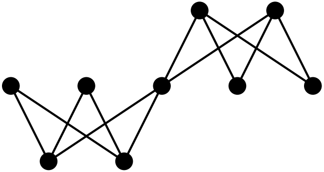

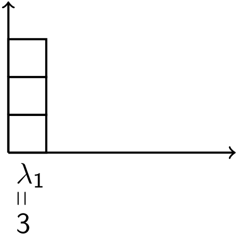

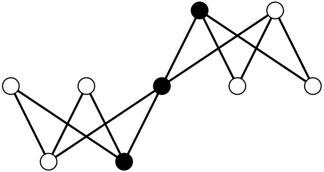

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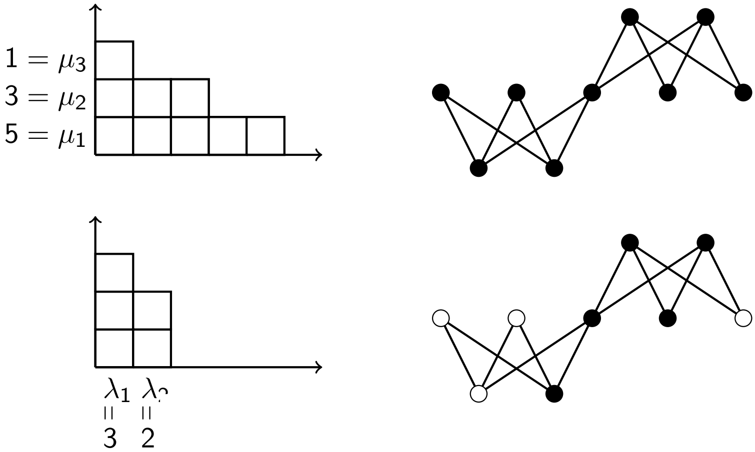

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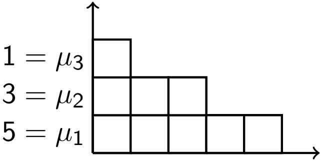

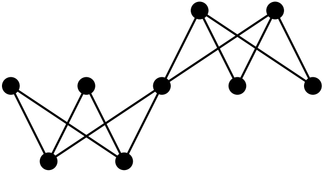

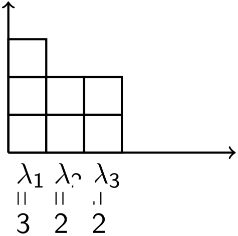

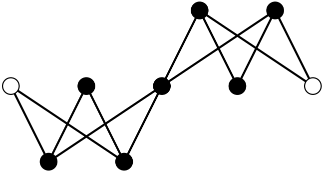

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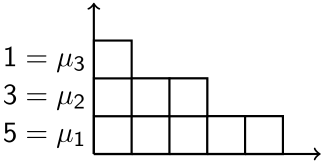

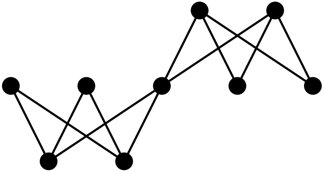

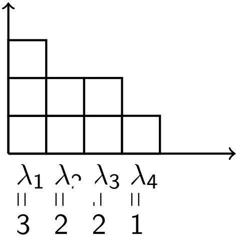

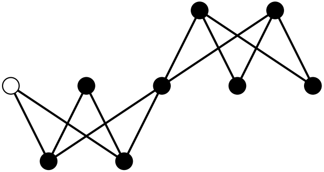

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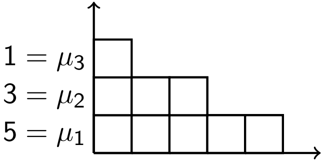
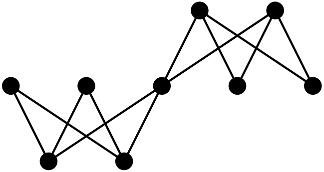
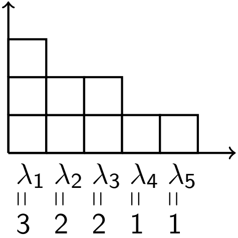
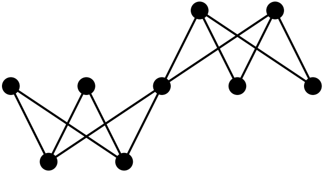




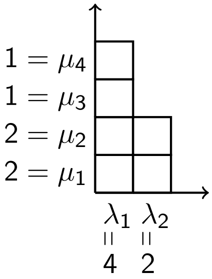


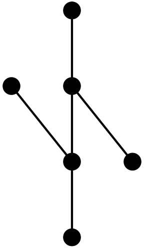


antichain-decomposablity and chain-decomposablity

Poset is antichain-decomposable if it has an antichain partition $A_1, \ldots, A_h$ with $|\bigcup_{i=1}^{k} A_i| = a_k = \sum_{i=1}^{k} \mu_i$.

Poset is chain-decomposable if it has a chain partition $C_1, \ldots, C_w$ with $|\bigcup_{i=1}^{j} C_i| = c_j = \sum_{i=1}^{j} \lambda_i$.





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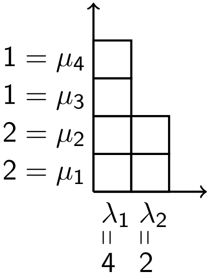

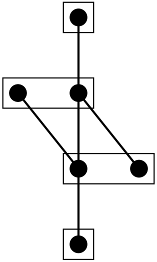

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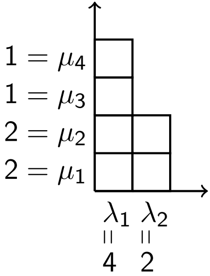

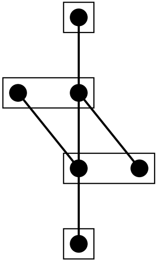

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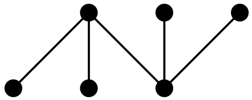

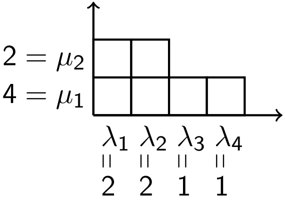

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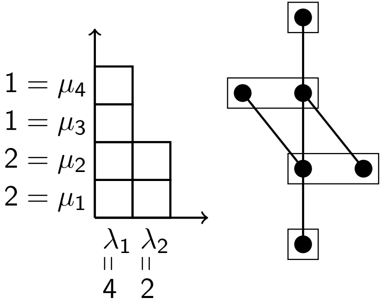

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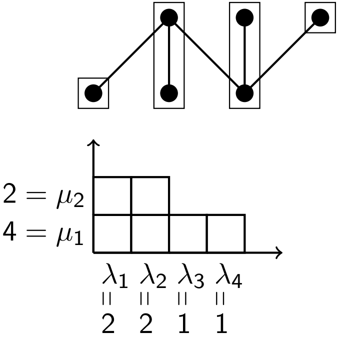

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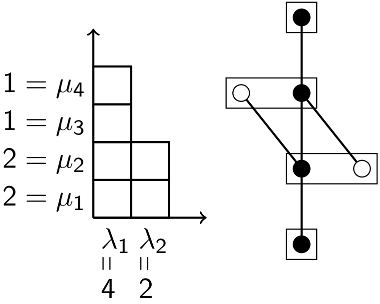

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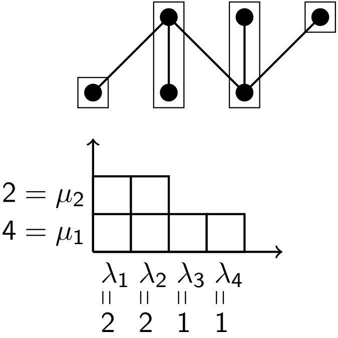

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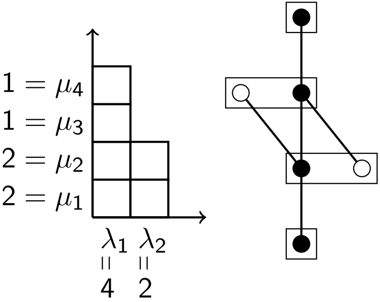

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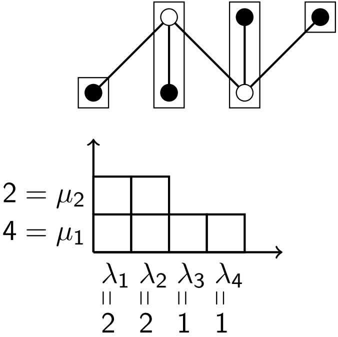

$|C_i| = \lambda_i$ for each i

Our results

Theorem

If P is antichain-decomposable and chain-decomposable and Q is antichain-decomposable, then Semiantichain Conjecture is satisfied for product $P \times Q$.

Remarks:

- Boolean latices are antichain- and chain-decomposable.
- Posets of width $\leqslant 3$ are antichain-decomposable.
- Serial-parallel posets are antichain- and chain-decomposable.
- there are more ...

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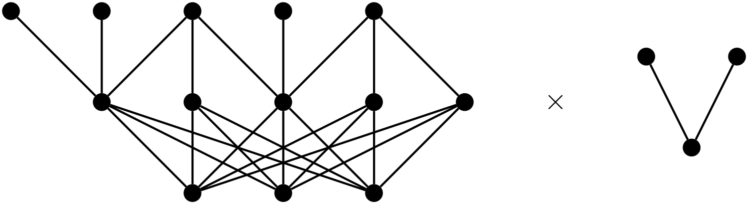
Counterexample in the general case

size of maximum semiantichain is 15
size of mininum unichain cover is 16



- Semiantichain Conjecture is NOT TRUE (after 30 years).
- The gap between maximum semiantichain and minimum unichain cover may be as large as we want.

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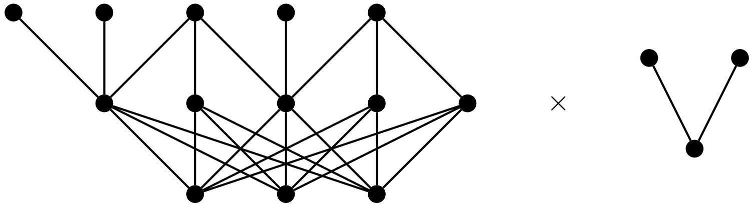
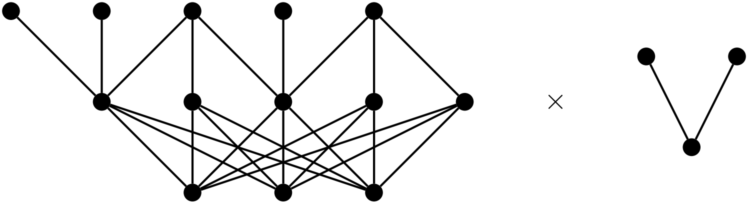
size of mininum unichain cover is 16



- Semiantichain Conjecture is NOT TRUE (after 30 years).
- The gap between maximum semiantichain and minimum unichain cover may be as large as we want.

Counterexample in the general case

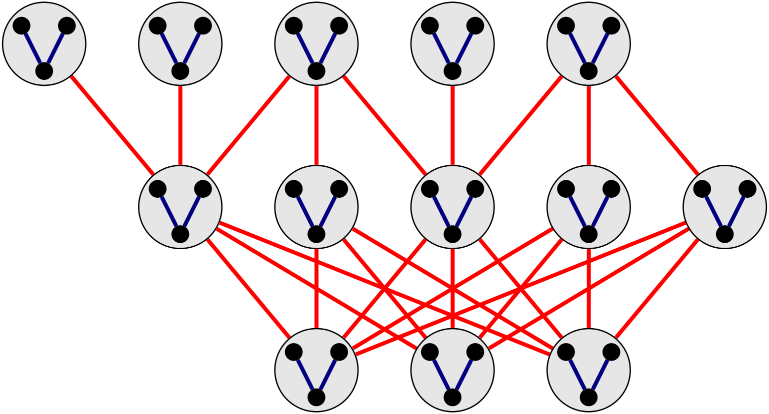
size of maximum semiantichain is 15
size of mininum unichain cover is 16



- Semiantichain Conjecture is NOT TRUE (after 30 years).
- The gap between maximum semiantichain and minimum unichain cover may be as large as we want.

Lemma

If P is a weak order and Q is an arbitrary poset, then the maximal size of a semiantichain in $P \times Q$ can be expressed as $\sum_{i=1}^{k} \mu_i^P \cdot |B_i|$ where $B_1, B_2, \ldots, B_k$ is a family of disjoint antichains in Q.



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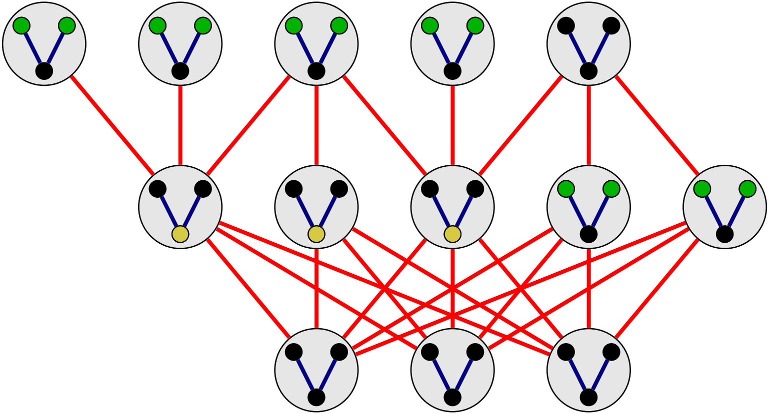

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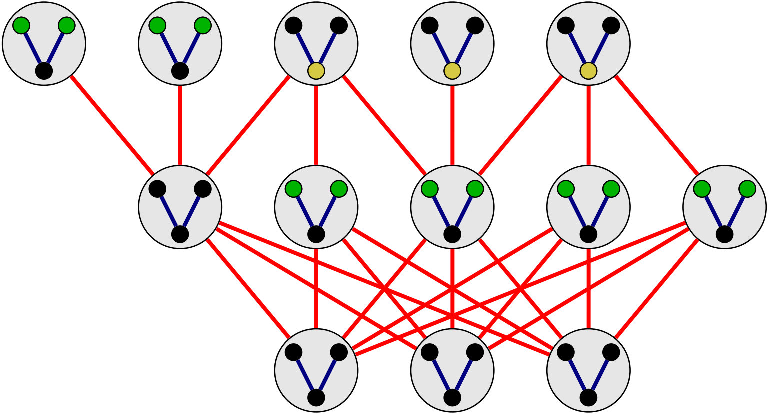
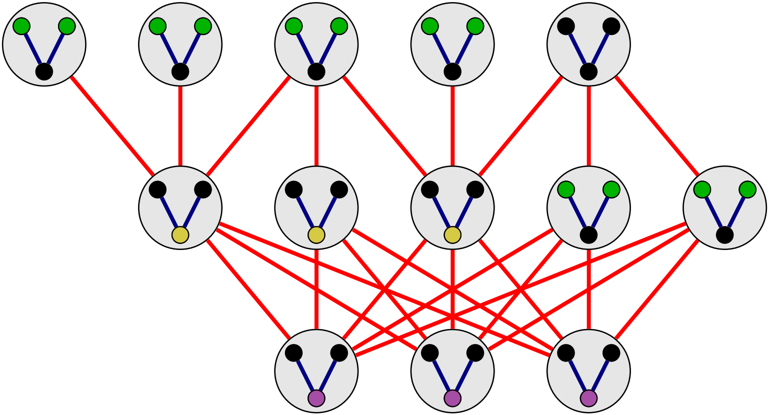
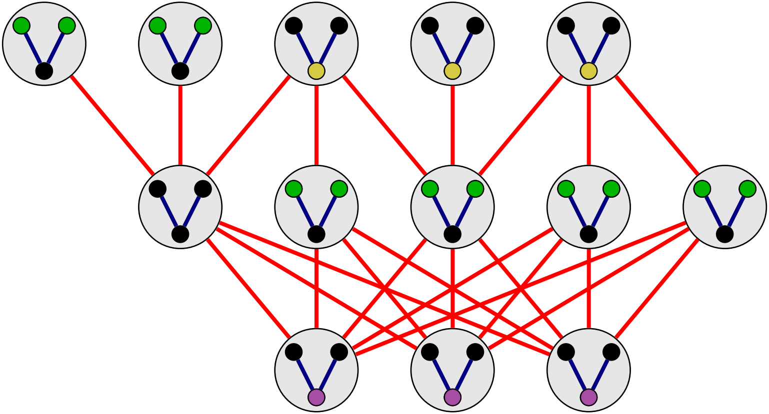


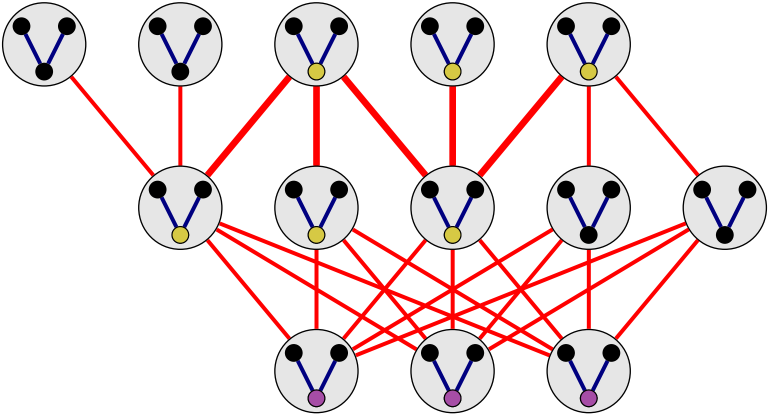
The proof



The proof



The proof



Open problem

Question

What is the complexity of deciding if product of two given posets satisfies Semiantichain Conjecture? Is it P or NPC?

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