Symmetric Chain Decompositions of Quotients of Chain Products by Wreath Products

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- P has a *symmetric chain decomposition* if $P = \bigsqcup_{i=0}^{n} C_i$ with each C_i a symmetric, saturated chain in P

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and for chains C_i of distinct lengths and $n_i \in \mathbb{N}$ $(i = 1, 2, \ldots, m)$

$$\mathrm{Aut}(C_1^{n_1} \times C_2^{n_2} \times \cdots \times C_m^{n_m}) \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}.$$

An Example

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$B_6/\langle(1\ 2\ 3\ 4\ 5\ 6)\rangle$

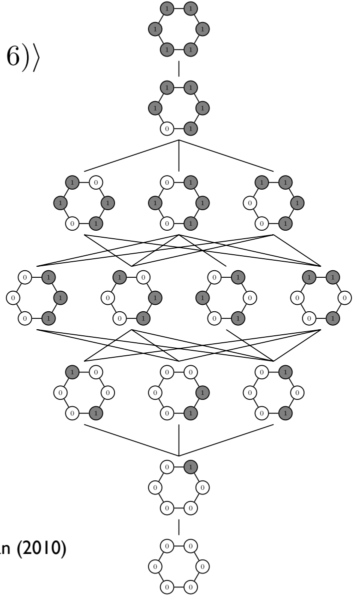

Figure credit: K K Jordan (2010)

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- for any $G \leq S_n$, $\mathbf{2}^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner;
- for any P that is a product of chains and $G \leq \mathrm{Aut}(P)$, P/G is rank-symmetric, rank-unimodal and strongly Sperner.

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Let P be a rank-symmetric partially ordered set of length n, say $P = \bigsqcup_{i=0}^{n} P_i$ and $r_i = |P_i|$ $(i = 0, 1, \ldots, n)$. Then P is rank-unimodal and strongly Sperner $\iff$ $\forall i = 0, 1, \ldots, \lfloor n/2 \rfloor$

$$\exists\ r_i \text{ pwd saturated chains } x_i < x_{i+1} < \cdots < x_{n-i}$$

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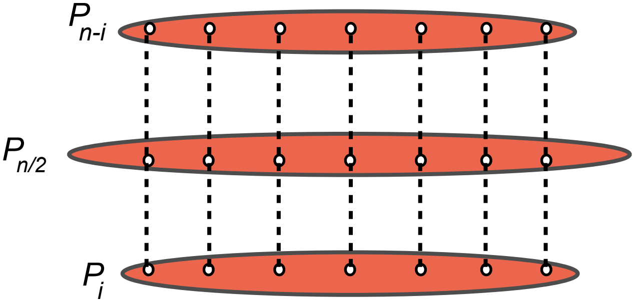

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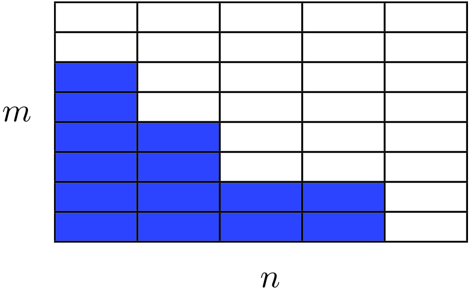

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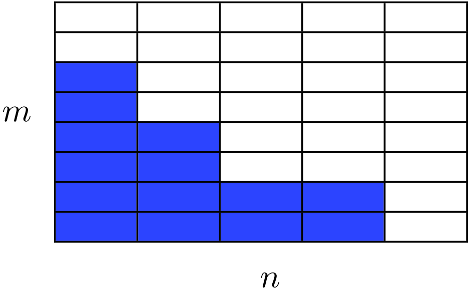

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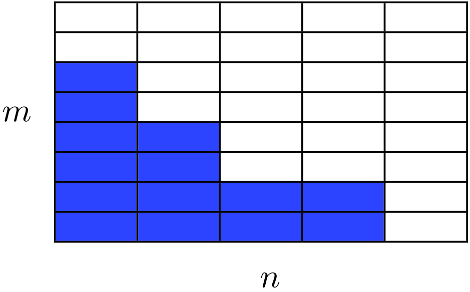

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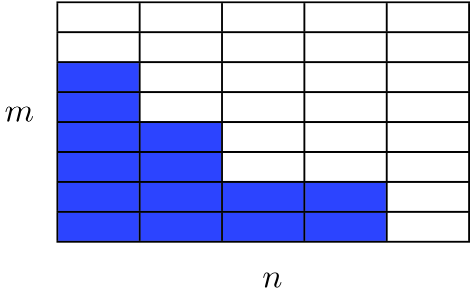

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4. for all n and all SCOs P, $P^n/\mathbb{Z}_n$ is an SCO [Dhand 2011].

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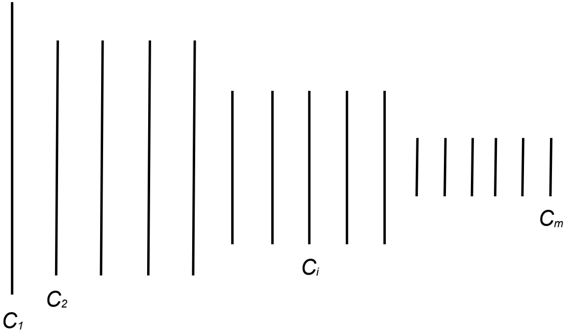

Fact: There is a subsequence $C'_{i_1}, C'_{i_2}, \ldots, C'_{i_s}$ of saturated, symmetric subchains that consist of exactly one representative of each orbit induced by $\sigma_i^{r_i}$:

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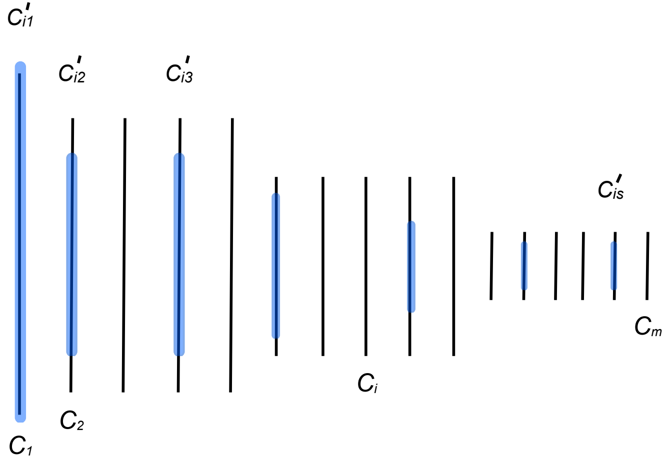

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Let $r = (k-1)q$, take the SCD $C'_{i_1}, C'_{i_2}, \ldots, C'_{i_s}$ of $\mathbf{2}^n/\langle \sigma^r \rangle$ obtained by pruning the GK SCD of $\mathbf{2}^n$ and select the subfamily of those $C'_{i_j} \subseteq C^m$.

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We want: for a chain C and $\phi = (1\ 2 \cdots m)$, $C^m/\langle\phi^q\rangle$ is an SCO.

Let C be the k-chain $000\ldots0, 100\ldots0, 110\ldots0, \ldots, 111\ldots1$ in $\mathbf{2}^{k-1}$.

With $n = (k-1)m$, C^m is the sublattice of $\mathbf{2}^n$ of all $0,1$-sequences of length n of the form $\mathbf{b} = \mathbf{b_1 b_2 \ldots b_m}$, $\mathbf{b}_i \in C$.

Fact: Each Greene-Kleitman chain C_i satisfies

$$C_i \cap C^m = \emptyset \text{ or } C_i \subseteq C^m.$$

With the embedding of C given above, we have $\phi^q = \sigma^{(k-1)q}{}_{|C^m}$.

Let $r = (k-1)q$, take the SCD $C'_{i_1}, C'_{i_2}, \ldots, C'_{i_s}$ of $\mathbf{2}^n/\langle\sigma^r\rangle$ obtained by pruning the GK SCD of $\mathbf{2}^n$ and select the subfamily of those $C'_{i_j} \subseteq C^m$.

Since $C^m/\langle\phi^q\rangle$ is a saturated and symmetric suborder of $\mathbf{2}^n/\langle\sigma^r\rangle$, these chains provide an SCD of $C^m/\langle\phi^q\rangle$.

A strategy for more general quotients $\mathbf{2}^n/G$

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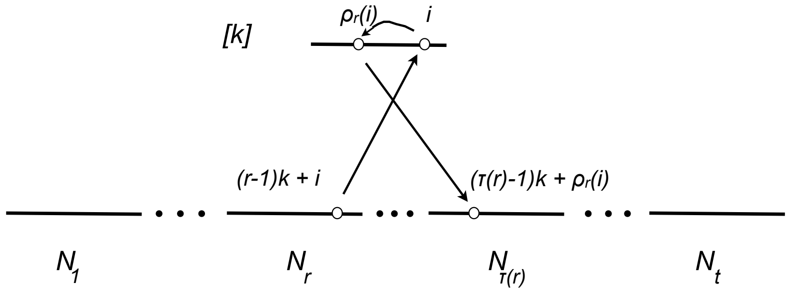
Partition $[n] = \bigcup_{r=1}^{t} N_r, \quad N_r = [(r-1)k+1, rk]$.

For each $\phi \in G$ there exist $\overline{\rho} = (\rho_1, \ldots, \rho_t) \in K^t$ and $\tau \in T$ such that

$$\phi((r-1)k+i) \ = \ (\tau(r)-1)k + \rho_r(i).$$



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Given an SCD $C_1, C_2, \ldots, C_s$ of $2^k/K$, take the shifted SCD $C_1^r, C_2^r, \ldots, C_s^r$ for each quotient $\mathbf{2}^{N_r}/K_r$:

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Each $C(\bar{j})$ is a symmetric, saturated suborder of $\mathbf{2}^n/K'$ and, since a chain product, $C(\bar{j})$ is an SCO. The collection of their SCDs provides an SCD for $\mathbf{2}^n/K'$.

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It suffices to select a representative K'-orbit from within each G-orbit such that the set of all selected orbits constitute symmetric, saturated suborders of the cubes $C(\bar{j})$. We will then have a copy of $\mathbf{2}^n/G$ along with an SCD.

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Fact 1: Each $\tau \in T$ has an action on $[s]^t$:

$$\tau(\bar{j}) = \tau(j_1, j_2, \ldots, j_t) = (j_{\tau^{-1}(1)}, j_{\tau^{-1}(2)}, \ldots, j_{\tau^{-1}(t)}).$$

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This induces a permutation $\widehat{\tau}$ on $\{C(\bar{j}) \mid \bar{j} \in [s]^t\}$ such that the restriction

$$\widehat{\tau} : C(\bar{j}) \mapsto C(\tau(\bar{j})) \quad \text{satisfies}$$

$$\widehat{\tau}([X_1], [X_2], \ldots, [X_t]) = ([X_{\tau^{-1}(1),1}], [X_{\tau^{-1}(2),2}], \ldots, [X_{\tau^{-1}(t),t}])$$

where $X_{r,q}$ is the shift of $X_r \subseteq N_r$ to N_q.

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Fact 2: Given $X, Y \subseteq [n]$ and $\overline{X} = ([X_1], [X_2], \ldots, [X_t]) \in C(\bar{j})$,

X and Y are in the same G-orbit $\iff$

there is some $\tau \in T$ such that $\overline{Y} \in C(\tau(\bar{j}))$ and $\widehat{\tau}(\overline{X}) = \overline{Y}$.

We see from Fact 2 that we should select a representative, say the lexicographically least, from each orbit of $[s]^t$ defined by T. Say J denotes the set of these representatives.

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In fact, for $X \subseteq [n]$ with $X = \bigcup_{r=1}^{t} X_r$ each X_r determines a unique $j'_r \in [s]$ via $[X_r] \in C^r_{j'_r}$ in the SCD of $2^{N_r}/K_r$. Thus, X belongs to a G-orbit intersecting a unique member of $\mathcal{C}_J$.

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However, a G-orbit may intersect some $C(\bar{j}) \in \mathcal{C}_J$ in more than one element. We must find a symmetric, saturated subset of such a cube $C(\bar{j})$ to insure that we have a unique representative of each G-orbit. In fact, we want that subset to have an SCD.

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Suppose that $t = 6$, $s = 5$, $T = \langle\sigma\rangle$ where $\sigma = (1\ 2 \cdots 6)$ and $\bar{j} = (3, 5, 3, 5, 3, 5)$.

Then for any $\tau \in T_{(3,5,3,5,3,5)} = \{1, \sigma^2, \sigma^4\}$, the stabilizer of $\bar{j}$, and any $\overline{X} \in C(\bar{j})$, $\overline{X}$ and $\widehat{\tau}(\overline{X})$ are in the same G-orbit.

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Suppose that $T = (\rho_1, \ldots, \rho_m) \leq S_t$ where the ρ_i's have disjoint support. Then for any $C(\bar{j})$, $T_{\bar{j}} = (\rho_1^{d_1}, \ldots, \rho_m^{d_m})$ where d_i is minimum such that the cycles of $\rho_i^{d_i}$ refine the coloring of $[t]$ defined by $\bar{j}$.

Theorem: Let $n = kt$, $G \leq S_n$, $K \leq S_k$, $T \leq S_t$, $G = K \wr T$ via the natural action of T on K^t. If both K and T are generated by powers of disjoint cycles then $\mathbf{2}^n/G$ is an SCO.

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Still no general argument for the dihedral group D_{2n} acting on $\mathbf{2}^n$