Variations on the Majorization Order

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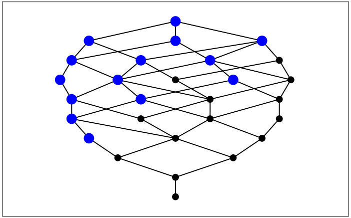
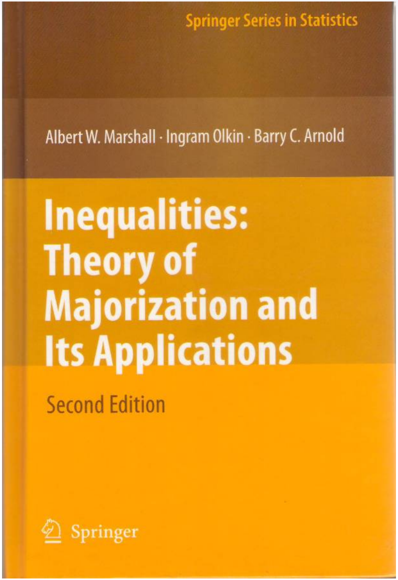
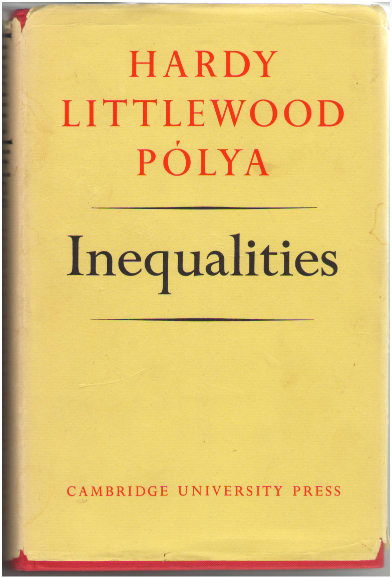
Halifax, June 18, 2012





A. W. Marshall, I. Olkin, B. C. Arnold
Inequalities: Theory of Majorization and its Applications, 2nd Edition
Springer 2010 (909 p).



G. H. Hardy, J. E. Littlewood, G. Polya
Inequalities
Cambridge U. Press 1934, 1951, 1967
(324 p).

Muirhead's Inequalities (1902)

Example:

$$\frac{1}{6}(A^2B^2 + A^2C^2 + \cdots + C^2D^2) \geq \frac{1}{12}(A^2BC + A^2BD + \cdots + B^2CD)$$

Denote this by:

$$M_{22} \gg M_{211}.$$

Similarly:

$$\frac{1}{24}(A^5B^4C + \cdots + B^5C^4D) \geq \frac{1}{12}(A^4B^4C^2 + \cdots + B^4C^2D^2)$$

Denote this by:

$$M_{541} \gg M_{442}.$$

Theorem (Muirhead): $M_\lambda \gg M_\mu \Leftrightarrow \lambda \succeq \mu$ ("Majorization")

Majorization: Definition

Assume λ and μ are monotone decreasing sequences. Then $\lambda \succeq \mu$ iff

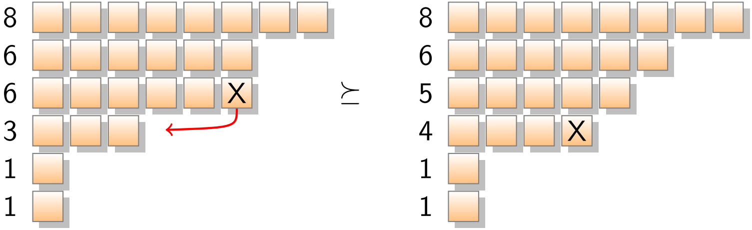
$$\begin{aligned} \lambda_1 &\geq \mu_1 \\ \lambda_1 + \lambda_2 &\geq \mu_1 + \mu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\geq \mu_1 + \mu_2 + \lambda_3 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &\geq \mu_1 + \mu_2 + \lambda_3 + \lambda_4 \\ &\vdots \quad \text{etc.} \end{aligned}$$

Example: $\lambda = \{8, 6, 6, 3, 1, 1\} \succeq \{8, 6, 5, 4, 1, 1\} = \mu$

Compare partial sums:

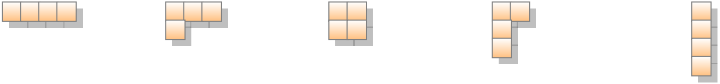
8	14	20	23	24	25	(λ)
8	14	19	23	24	25	(μ)

Moving Boxes



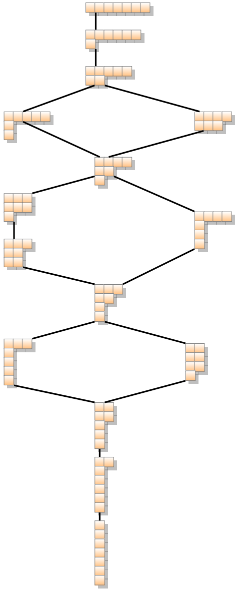
Example:

$$(4,0,0,0) \succeq (3,1,0,0) \succeq (2,2,0,0) \succeq (2,1,1,0) \succeq (1,1,1,1)$$



$$M_4 \gg M_{31} \gg M_{22} \gg M_{211} \gg M_{1111}$$

The Poset $(P_6, \preceq)$: Partitions of 6



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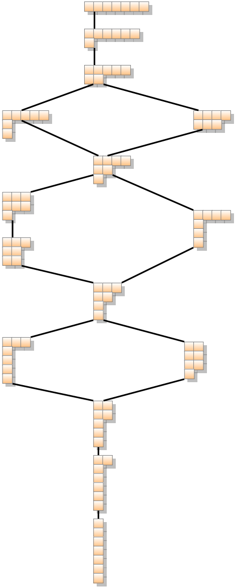
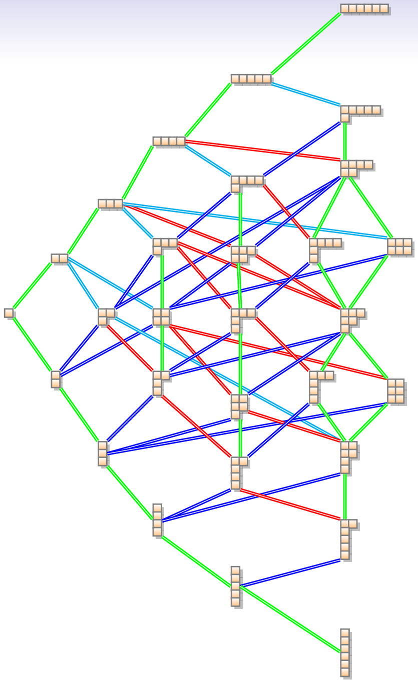


Properties

- Lattice
- Self-dual $(\lambda \leftrightarrow \lambda^\top)$
- Möbius function $\pm 1, 0$

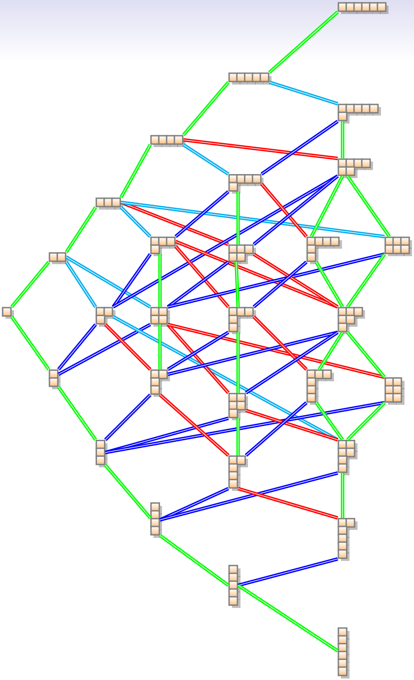
Non-Properties

- Not ranked



The Poset $DP_{\leq 6}$

"Double-Majorization"



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- What is it?
- Why interesting?
- Properties?

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Non-homogeneous Muirhead-type inequalities

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$$\frac{1}{6}(AB+AC+AD+BC+BD+CD) < ? > \frac{1}{4}(ABC+ABD+ACD+BCD)$$

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These last inequalities are TRUE (MacLaurin (1729)).

"Symmetric Monomial Means"

$$\left(\frac{1}{6}(AB + AC + \cdots + BD + CD)\right)^{1/2} = \mathfrak{M}_{11}$$

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Theorems: $\mathfrak{M}_{11} \gg \mathfrak{M}_{111}$ (1729), $\mathfrak{M}_{33} \gg \mathfrak{M}_{211}$ (2009).

What is it?

"Normalized majorization"

NORMALIZE: $\lambda \longmapsto \bar{\lambda} = \frac{\lambda}{|\lambda|}$

Embeds each $\mathcal{P}_n$ naturally into the lattice $(\mathcal{Q}_1, \preceq)$ of nonnegative monotone rational sequences summing to 1, under majorization.

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$$(4,\ 3,\ 1,\ 1,\ 1) \sqsubseteq (3,\ 3,\ 0,\ 0,\ 0)$$
$$(.5, .7, .8, .9, 1) \preceq (.5,\ 1,\ 1,\ 1,\ 1)$$

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But note that $(1,1,1) \sqsubseteq (2,2,2) \sqsubseteq (1,1,1)$. (It's a preorder.)

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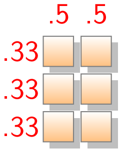
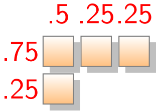
Example: $222 \trianglelefteq 31$

(Percentages)



$\trianglelefteq$



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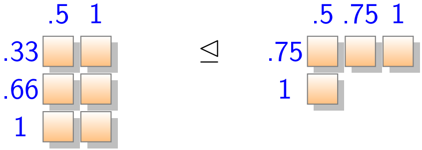

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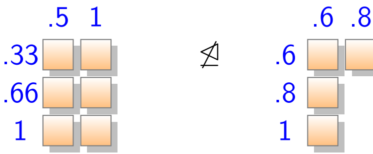
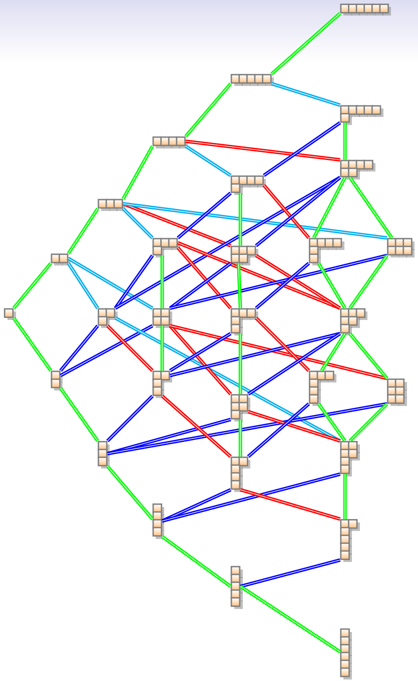
(Partial sums)

Example: $222 \not\trianglelefteq 311$



Observe that the conditions $\lambda \sqsubseteq \mu$ and $\lambda^\top \sqsupseteq \mu^\top$ are not equivalent.



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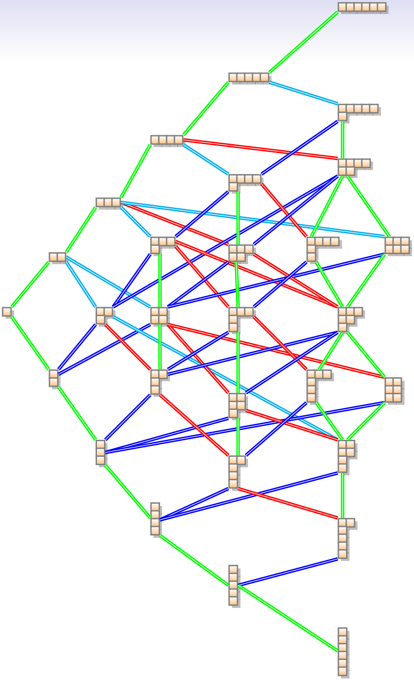

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Application: Muirhead for Symmetric Monomial Means

Theorem (Cuttler-Greene-Skandera 2011) If λ and μ are partitions with $|\lambda| \leq |\mu|$, then $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ if and only if $\lambda \trianglelefteq \mu$.

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- True for all cases shown in the diagram.
- Includes several families of classical inequalities (e.g. Maclaurin 1729).
- Actually, a very strong form of the inequality holds ("y-positivity").
- "Only if" part true for all λ, μ.

Properties of $\mathcal{D}P_\infty$

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- $\mathcal{D}P_\infty$ is an infinite poset without universal bounds; it is locally finite, but is not locally ranked.
- All coverings are obtained by adding or moving boxes up, but not always a single box.

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- All such coverings are obtained as follows: if $|\lambda| = a$, $|\mu| = b$ with $a < b$, then:
  - Let $S(\lambda)$ denote the sequence of partial sums of λ, and let $S^b(\lambda) = \lceil (b/a)S(\lambda) \rceil$. Let μ_0 be the unique composition whose partial sum sequence is $S^b(\lambda)$.

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    - Then μ is the unique partition obtained from μ_0 obtained by repeatedly applying $(\ldots \mu_i \mu_{i+1} \ldots) \mapsto (\ldots \mu_{i+1} \mu_i \ldots)$ if $\mu_i < \mu_{i+1}$.

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- If $|\lambda| = a, |\mu| = b, a < b$ and λ covers μ, then μ is (uniquely) obtained by applying the above algorithm to $\lambda^\top$.

Coverings in $\mathcal{D}P_\infty$

Example: $\lambda = (1, 1, 1, 1), a = 4, b = 6.$

$$\begin{aligned} S(\lambda) &= (1, 2, 3, 4) \\ (6/4)S(\lambda) &= (3/2, 3, 9/2, 6) \\ S^6(\lambda) &= (2, 3, 5, 6) \\ \mu_0 &= (2, 1, 2, 1) \\ \mu &= (2, 2, 1, 1) \end{aligned}$$

Conclusion: λ is covered by μ.

Variation: Majorization on Posets

Setup: P = finite poset, $\mathbb{Z}[P]$ = set maps from P to $\mathbb{Z}$. Identify $f \in \mathbb{Z}[P]$ with the formal sum $\sum_{x \in P} f(x)x$.

Problem: Given $f \in \mathbb{Z}[P]$, find conditions under which f can be written as a positive linear combination

$$\sum_{x<y} c_{xy}(y-x), \quad \text{with } c_{xy} \geq 0 \ \ \forall x<y.$$

Denote the set of such f's by $\mathcal{M}(P)$, and call $\mathcal{M}(P)$ the Muirhead cone of P,

Solution: $f \in \mathcal{M}(P)$ iff $f[K] \geq 0$ for all dual order ideals $K \subseteq P$ and $f[P] = 0$.

Definition (Majorization): $f \preceq g$ iff $g - f \in \mathcal{M}(P)$.

Some Properties (and Non-Properties)

Definition: $\mathbb{Z}_n[P] = \{f \in \mathbb{Z}[P] \mid |f| = n\}$
$\pi_n[P] = \{f \in \mathbb{Z}_n[P] \mid f \text{ is order-preserving}\}$

Both $(\mathbb{Z}_n[P], \preceq)$ and $(\pi_n[P], \preceq)$ form posets under majorization. The latter are "reverse P-partitions".

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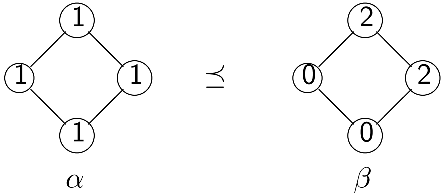
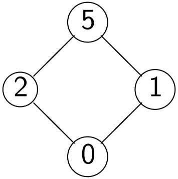
Theorem:

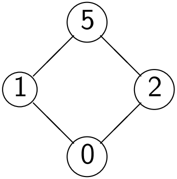
- $\mathbb{Z}_n[P]$ is ranked and self-dual, but in general it is not a lattice.
- In general, $\pi_n(P)$ is neither ranked nor self-dual, and it is not a lattice.
- Coverings in both $\mathbb{Z}_n[P]$ and $\pi_n[P]$ always consist of "moving boxes up". In $\mathbb{Z}[P]$ it is always a single box, but in $\pi_n[P]$ more than one box may be required.



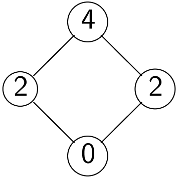
$\pi_n(P)$ is not a Lattice



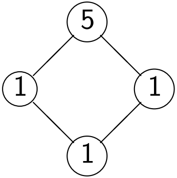
α



β



γ



δ

Another Variation: Principal Majorization

If we replace dual order ideals by principal dual order ideals, we get a larger cone

$$\mathcal{M}^+(P) = \{f \mid f[J] \geq 0 \text{ for all principal dual order ideals } J\},$$

and a new type of majorization:

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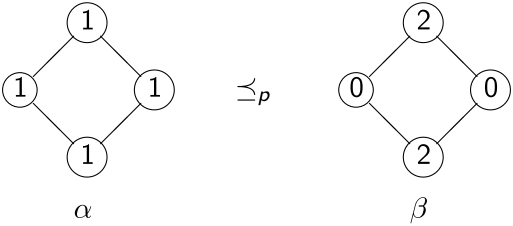
$$\mathcal{M}^+(P) = \{f \mid f[J] \geq 0 \text{ for all principal dual order ideals } J\},$$

and a new type of majorization:

Definition: $f \preceq_p g$ iff $(g - f)[J] \geq 0$ for all principal dual order ideals J.



Extreme ray description of the cone $\mathcal{M}^+(P)$

Theorem: $f \in \mathcal{M}^+(P)$ iff f can be expressed as a positive linear combination

$$\sum_{z \in P} c_z \, \Delta_z, \quad \text{with } c_z \geq 0 \; \forall z,$$

where for all z,

$$\Delta_z = \sum_{y \leq z} \mu(y, z) y.$$

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Note: The cone $\mathcal{M}^+(P)$ contains the cone $\mathcal{M}(P)$. It's extreme generators $y - x$ are nonnegative linear combinations of the Δ_z's. (Standard Möbius function argument.)

References/Acknowledgements

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