Variations on the Majorization Order

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*Inequalities: Theory of Majorization and its Applications, 2nd Edition*

Springer 2010 (909 p.).
G. H. Hardy, J. E. Littlewood, G. Polya

_Inequalities_
Muirhead’s Inequalities (1902)

Example:

\[ \frac{1}{6}(A^2B^2 + A^2C^2 + \cdots + C^2D^2) \geq \frac{1}{12}(A^2BC + A^2BD + \cdots + B^2CD) \]

Denote this by:

\[ M_{22} \gg M_{211}. \]

Similarly:

\[ \frac{1}{24}(A^5B^4C + \cdots + B^5C^4D) \geq \frac{1}{12}(A^4B^4C^2 + \cdots + B^4C^2D^2) \]

Denote this by:

\[ M_{541} \gg M_{442}. \]

Theorem (Muirhead): \( M_{\lambda} \gg M_{\mu} \iff \lambda \succeq \mu \) ("Majorization")
Majorization: Definition

Assume $\lambda$ and $\mu$ are monotone decreasing sequences. Then $\lambda \succeq \mu$ iff

\[
\begin{align*}
\lambda_1 & \geq \mu_1 \\
\lambda_1 + \lambda_2 & \geq \mu_1 + \mu_2 \\
\lambda_1 + \lambda_2 + \lambda_3 & \geq \mu_1 + \mu_2 + \lambda_3 \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & \geq \mu_1 + \mu_2 + \lambda_3 + \lambda_4 \\
\vdots & \quad \text{etc.}
\end{align*}
\]

Example: $\lambda = \{8, 6, 6, 3, 1, 1\} \succeq \{8, 6, 5, 4, 1, 1\} = \mu$

Compare partial sums: 8 14 20 23 24 25  (\(\lambda\))

8 14 19 23 24 25  (\(\mu\))
Moving Boxes

Example:

\[(4, 0, 0, 0) \succeq (3, 1, 0, 0) \succeq (2, 2, 0, 0) \succeq (2, 1, 1, 0) \succeq (1, 1, 1, 1)\]
The Poset \((P_6, \preceq)\): Partitions of 6

Properties
• Lattice
• Self-dual (\(\lambda \leftrightarrow \lambda^\top\))
• Möbius function \(\pm 1, 0\)

Non-Properties
• Not ranked
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"Double-Majorization"
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- What is it?
- Why interesting?
- Properties?
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Non-homogeneous Muirhead-type inequalities
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Non-homogeneous Muirhead-type inequalities

Examples:

\[
\frac{1}{6}(AB + AC + AD + BC + BD + CD) < ? > \frac{1}{4}(ABC + ABD + ACD + BCD)
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There can’t be an inequality, in either direction.
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\[
\left(\frac{1}{6}(AB+AC+\cdots+BD+CD)\right)^{1/2} \ \geq \ \left(\frac{1}{4}(ABC+ABD+ACD+BCD)\right)^{1/3}
\]
or

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\left(\frac{1}{6}(AB+AC+\cdots+BD+CD)\right)^{3} \ \geq \ \left(\frac{1}{4}(ABC+ABD+ACD+BCD)\right)^{2}
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These last inequalities are TRUE (MacLaurin (1729)).
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“Symmetric Monomial Means”

\[
\left( \frac{1}{6} (AB + AC + \cdots + BD + CD) \right)^{1/2} = M_{11}
\]

\[
\left( \frac{1}{4} (ABC + ABD + ACD + BCD) \right)^{1/3} = M_{111}
\]

\[
\left( \frac{1}{6} (A^3B^3 + A^3C^3 + B^3C^3 + A^3D^3 + B^3D^3 + C^3D^3) \right)^{1/6} = M_{33}
\]

\[
\left( \frac{1}{12} (A^2BC + A^2BD + A^2CD + \cdots + D^2AC + D^2BC) \right)^{1/4} = M_{211}
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Theorems: \( M_{11} \gg M_{111} \) (1729), \( M_{33} \gg M_{211} \) (2009).
What is it?
“Normalized majorization”

NORMALIZE: \[ \lambda \mapsto \bar{\lambda} = \frac{\lambda}{|\lambda|} \]

Embeds each \( P_n \) naturally into the lattice \( (Q_1, \preceq) \) of nonnegative monotone rational sequences summing to 1, under majorization.
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**Definition:**

\[ \lambda \sqsubseteq \mu \text{ if } \bar{\lambda} \preceq \bar{\mu}, \text{ i.e., } \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}. \]

**Example:**

\[
(4, 3, 1, 1, 1) \sqsubseteq (3, 3, 0, 0, 0) \\
(.5, .7, .8, .9, 1) \preceq (.5, 1, 1, 1, 1)
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\[ (.5, .7, .8, .9, 1) \preceq (.5, 1, 1, 1, 1) \]

But note that \( (1, 1, 1) \sqsubseteq (2, 2, 2) \sqsubseteq (1, 1, 1) \). (It’s a preorder.)
<table>
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**Example:** \( 222 \trianglelefteq 31 \)

\[
\begin{array}{c|c|c|c|c|c}
\text{(Percentages)} & .5 & .5 & \trianglelefteq & .5 & .25\ 25 \\
.33 & .33 & .75 & .25 \\
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\end{array}
\]
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**Example:** 222 \(\preceq\) 31

\[
\begin{array}{cccc}
.5 & 1 & \preceq & .5 \ .75 \ 1 \\
.33 & .66 & \preceq \preceq & \ .75 \\
1 & 1 & \preceq & 1
\end{array}
\]

(Partial sums)
“Double Majorization”

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Example: \( 222 \npreceq 311 \)

Observe that the conditions \( \lambda \subseteq \mu \) and \( \lambda^\top \supseteq \mu^\top \) are not equivalent.
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Application: Muirhead for Symmetric Monomial Means

**Theorem (Cuttler-Greene-Skandera 2011)** If \( \lambda \) and \( \mu \) are partitions with \(|\lambda| \leq |\mu|\), then \( M_\lambda \leq M_\mu \) if and only if \( \lambda \preceq \mu \).
**Application: Muirhead for Symmetric Monomial Means**

**Theorem (Cuttler-Greene-Skandera 2011)** If $\lambda$ and $\mu$ are partitions with $|\lambda| \leq |\mu|$, then $M_\lambda \leq M_\mu$ if and only if $\lambda \preceq \mu$.

**Conjecture** Also true when $|\lambda| > |\mu|$.
**Application: Muirhead for Symmetric Monomial Means**

**Theorem (Cuttler-Greene-Skandera 2011)** If \( \lambda \) and \( \mu \) are partitions with \( |\lambda| \leq |\mu| \), then \( M_\lambda \leq M_\mu \) if and only if \( \lambda \trianglelefteq \mu \).

**Conjecture** Also true when \( |\lambda| > |\mu| \).

- True for all cases shown in the diagram.
- Includes several families of classical inequalities (e.g. Maclaurin 1729).
- Actually, a very strong form of the inequality holds (“y-positivity”).
- “Only if” part true for all \( \lambda, \mu \).
Properties of $\mathcal{DP}_\infty$

- If $\lambda \preceq \mu$ and $\mu \preceq \lambda$, then $\lambda = \mu$; hence $\mathcal{DP}_\infty$ is a partial order.
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- $\lambda \trianglelefteq \mu$ if and only if $\lambda^\top \trianglerighteq \mu^\top$; hence $\mathcal{DP}_\infty$ is self-dual.
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- $\mathcal{DP}_\infty$ is an infinite poset without universal bounds; it is locally finite, but is not locally ranked.
- All coverings are obtained by adding or moving boxes up, but not always a single box.
Coverings in $\mathcal{DP}_\infty$

- If $|\lambda| = |\mu|$, then $\lambda$ is covered by $\mu$ if and only if $\mu$ is obtained from $\lambda$ by moving a single box up.
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- If $\lambda$ is covered by $\mu$ with $|\lambda| < |\mu|$, then $|\mu| \leq 2|\lambda|$. 
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- All such coverings are obtained as follows: if $|\lambda| = a$, $|\mu| = b$ with $a < b$, then:
  - Let $S(\lambda)$ denote the sequence of partial sums of $\lambda$, and let $S^b(\lambda) = \lceil (b/a)S(\lambda) \rceil$. Let $\mu_0$ be the unique composition whose partial sum sequence is $S^b(\lambda)$. 

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  - Then $\mu$ is the unique partition obtained from $\mu_0$ obtained by repeatedly applying $(\ldots \mu_i \mu_{i+1} \ldots) \mapsto (\ldots \mu_{i+1} \mu_i \ldots)$ if $\mu_i < \mu_{i+1}$. 
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- If $|\lambda| = a$, $|\mu| = b$, $a < b$ and $\lambda$ covers $\mu$, then $\mu$ is (uniquely) obtained by applying the above algorithm to $\lambda^\top$. 
Coverings in $\mathcal{DP}_\infty$

Example: $\lambda = (1, 1, 1, 1)$, $a = 4$, $b = 6$.

$$S(\lambda) = (1, 2, 3, 4)$$
$$\left(\frac{6}{4}\right) S(\lambda) = (3/2, 3, 9/2, 6)$$
$$S^6(\lambda) = (2, 3, 5, 6)$$
$$\mu_0 = (2, 1, 2, 1)$$
$$\mu = (2, 2, 1, 1)$$

Conclusion: $\lambda$ is covered by $\mu$. 
Variation: Majorization on Posets

Setup: $P = \text{finite poset}$, $\mathbb{Z}[P] = \text{set maps from } P \text{ to } \mathbb{Z}$. Identify $f \in \mathbb{Z}[P]$ with the formal sum $\sum_{x \in P} f(x)x$.

Problem: Given $f \in \mathbb{Z}[P]$, find conditions under which $f$ can be written as a positive linear combination

$$\sum_{x < y} c_{xy}(y - x), \quad \text{with } c_{xy} \geq 0 \quad \forall x < y.$$ 

Denote the set of such $f$’s by $\mathcal{M}(P)$, and call $\mathcal{M}(P)$ the Muirhead cone of $P$.

Solution: $f \in \mathcal{M}(P)$ iff $f[K] \geq 0$ for all dual order ideals $K \subseteq P$ and $f[P] = 0$.

Definition (Majorization): $f \preceq g$ iff $g - f \in \mathcal{M}(P)$. 
Some Properties (and Non-Properties)

Definition: \( \mathbb{Z}_n[P] = \{ f \in \mathbb{Z}[P] \mid |f| = n \} \)
\( \pi_n[P] = \{ f \in \mathbb{Z}_n[P] \mid f \text{ is order-preserving} \} \)

Both \( (\mathbb{Z}_n[P], \preceq) \) and \( (\pi_n[P], \preceq) \) form posets under majorization. The latter are “reverse \( P \)-partitions”. 
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Theorem:

- $\mathbb{Z}_n[P]$ is ranked and self-dual, but in general it is not a lattice.
- In general, $\pi_n(P)$ is neither ranked nor self-dual, and it is not a lattice.
- Coverings in both $\mathbb{Z}_n[P]$ and $\pi_n[P]$ always consist of “moving boxes up”. In $\mathbb{Z}[P]$ it is always a single box, but in $\pi_n[P]$ more than one box may be required.

\[
\begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \quad \prec \quad \begin{array}{c}
0 \\
0 \\
2 \\
\end{array}
\]

$\alpha$ \quad $\beta$
\( \pi_n(P) \) is not a Lattice

\[ \begin{array}{cccc}
2 & 1 & 0 & 4 \\
5 & 1 & 0 & 2 \\
2 & 4 & 0 & 2 \\
0 & 5 & 1 & 1 \\
\end{array} \]
Another Variation: Principal Majorization

If we replace dual order ideals by principal dual order ideals, we get a larger cone

$$\mathcal{M}^+(P) = \{ f | f[J] \geq 0 \text{ for all principal dual order ideals } J \},$$

and a new type of majorization:

**Definition:** $f \preceq_p g$ iff $(g - f)[J] \geq 0$ for all principal dual order ideals $J$. 
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\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\preceq_p
\begin{array}{ccc}
2 & 2 & 2 \\
0 & 0 & 0 \\
2 & 2 & 2 \\
\end{array}
\]

\( \alpha \)

\( \beta \)
Extreme ray description of the cone $\mathcal{M}^+(P)$

**Theorem:** $f \in \mathcal{M}^+(P)$ iff $f$ can be expressed as a positive linear combination

$$\sum_{z \in P} c_z \Delta_z, \quad \text{with } c_z \geq 0 \ \forall z,$$

where for all $z$,

$$\Delta_z = \sum_{y \leq z} \mu(y, z)y.$$
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$$
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$$

Note: The cone $\mathcal{M}^+(P)$ contains the cone $\mathcal{M}(P)$. It’s extreme generators $y - x$ are nonnegative linear combinations of the $\Delta_z$’s. (Standard Möbius function argument.)
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