An Improved Bound for First-Fit on Posets Without Two Long Incomparable Chains

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First-Fit coloring of graphs

Colors $=$ positive integers

First-Fit coloring of G:

- pick an ordering of the vertices, and
- color each vertex in order with the smallest available color

Remark: First-Fit coloring $\Leftrightarrow$ coloring in which every vertex colored i has neighbors colored $1, 2, \ldots, i-1$

$\mathrm{FF}(G)$:= max. number of colors in a First-Fit coloring

First-Fit on interval graphs

Interval graphs:

Much studied question: FF(G) vs $\omega(G)$ when G interval graph

FF(G) $\leqslant$	Authors
$40\omega(G)$	Kierstead
$25.8\omega(G)$	Kierstead & Qin
$8\omega(G)$	Pemmaraju, Raman and Varadarajan (*)

Current record for lower bounds:

$\forall\varepsilon > 0\ \exists G$ s.t. FF(G) $> (5-\varepsilon)\omega(G)$ (Smith, 2010)

First-Fit chain partitioning of posets

First-Fit chain partitioning of P:

- pick an ordering of the elements, and
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Same as First-Fit coloring of incomparability graph G of P

Note: $\omega(G) =$ width of P

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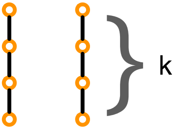
Interval orders:

G interval graph $\Leftrightarrow$ G incomparability graph of an interval order

Posets without $\mathbf{k} + \mathbf{k}$



forbidden as induced subposet

Interval orders = posets without $\mathbf{2} + \mathbf{2}$ (Fishburn)

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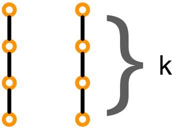

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Interval orders = posets without $\mathbf{2} + \mathbf{2}$ (Fishburn)

Let P be a poset of width w without $\mathbf{k} + \mathbf{k}$

First-Fit uses at most

- $3kw^2$ chains (Bosek, Krawczyk, and Szczypka)

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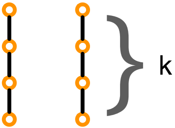

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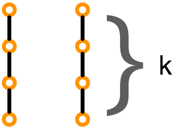

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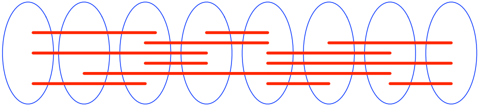
- $16kw$ chains (this talk)

Pathwidth

- Path decomposition of a graph G: sequence $B_1, \ldots, B_q$ of subsets of $V(G)$ (called bags) s.t.
  - for every edge uv there exists a bag containing both u and v
  - every vertex appears in a non-empty set of consecutive bags



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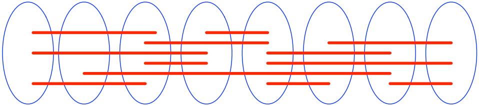

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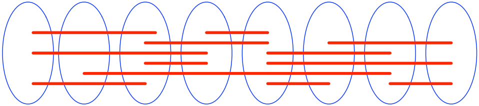

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Equivalently,

$\mathrm{pw}(G) \leqslant t \Leftrightarrow G \subseteq H$ for some interval graph H with $\omega(H) \leqslant t+1$

homomorphism from G to H: function $f : V(G) \to V(H)$ that maps edges of G to edges of H

Theorem (Dujmović, J., Wood)

Every graph G with $\mathrm{pw}(G) \leqslant p$ is homomorphic to an interval graph H with $\omega(H) \leqslant p + 1$ and $\mathrm{FF}(G) \leqslant \mathrm{FF}(H)$.

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Every graph G with $\mathrm{pw}(G) \leqslant p$ *is homomorphic to an interval graph H with* $\omega(H) \leqslant p + 1$ *and* $\mathrm{FF}(G) \leqslant \mathrm{FF}(H)$.

Remark: implicitly shown by Kierstead and Saoub (*First-Fit coloring of bounded tolerance graphs*. Discrete Applied Mathematics, 2011)

Corollary

$\mathrm{FF}(G) \leqslant 8(\mathrm{pw}(G) + 1)$ *for every graph G*

Theorem (DJW)

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Proof:

$K :=$ spanning interval supergraph of G with $\omega(K) \leqslant p + 1$

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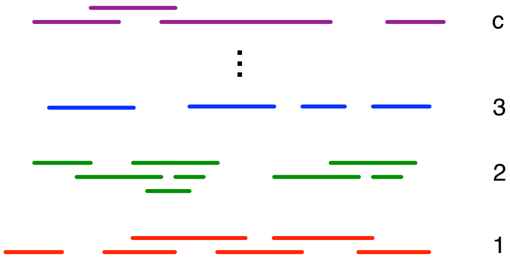
Consider a First-Fit coloring of G *in the graph* K:



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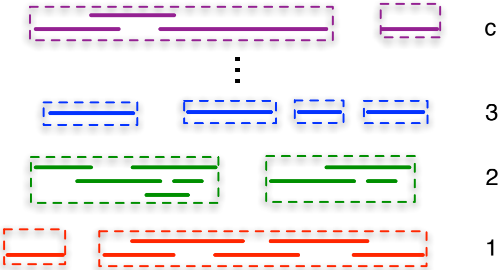

$\rightarrow$ graph H

Pathwidth of incomparability graphs

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G incomparability graph of P

Theorem (Dujmović, J., Wood)

$\mathrm{pw}(G) \leqslant 2kw - 1$

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Theorem (Dujmović, J., Wood)

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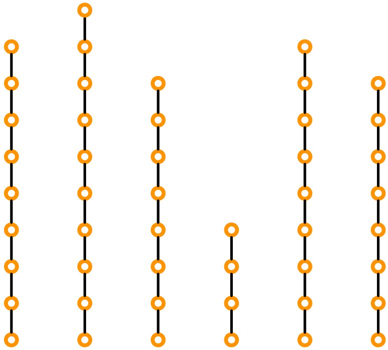
Remarks:

- upper bound can be improved to $(2k-3)w - 1$
- $\exists P$ such that $\mathrm{pw}(G) \geqslant (k-1)(w-1)$

Sketch of proof

Dilworth chain decomposition $C_1, \ldots, C_w$ of P



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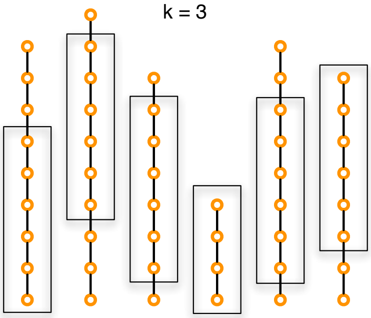

X **block** if $X \cap C_i$ consists of $\min\{2k, |C_i|\}$ consecutive elements of C_i $\forall i$

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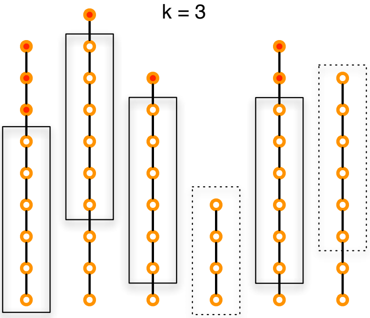

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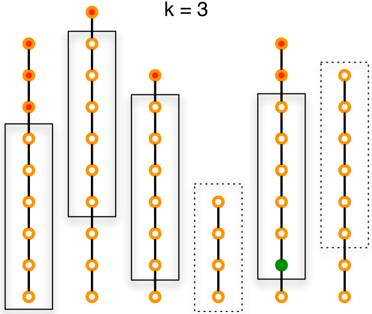
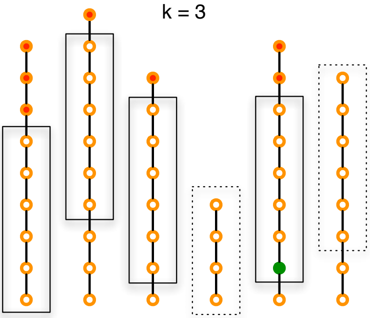
up(X) chain C_i *alive* or *dead*

Sketch of proof



$u \in X \cap C_i$ **good** if

- C_i alive,
- u minimal in $X \cap C_i$,
- $u < v \quad \forall v \in \mathrm{up}(X)$



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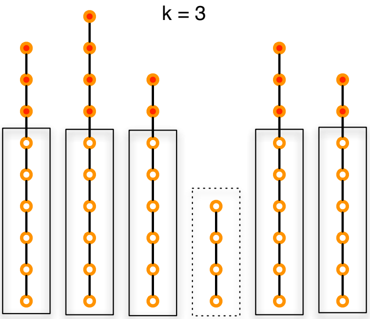

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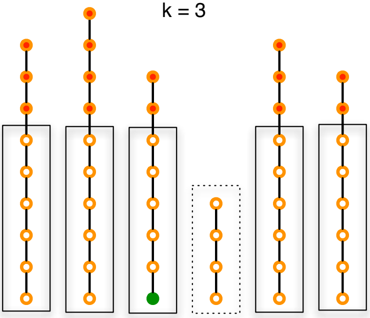

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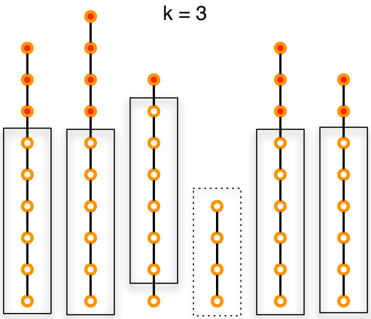

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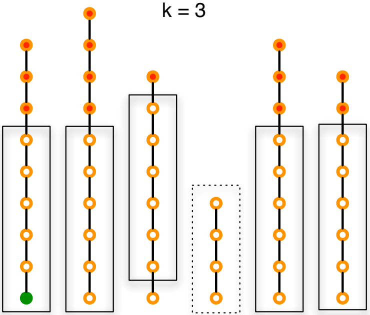

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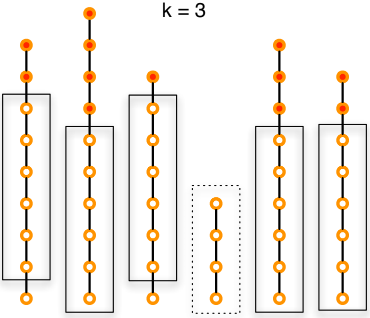

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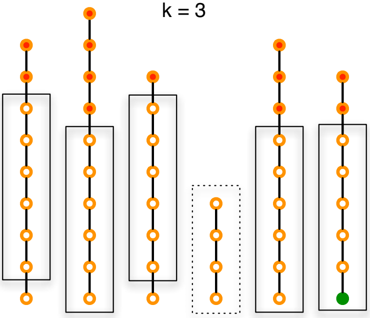

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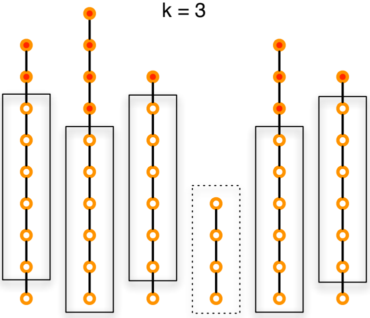

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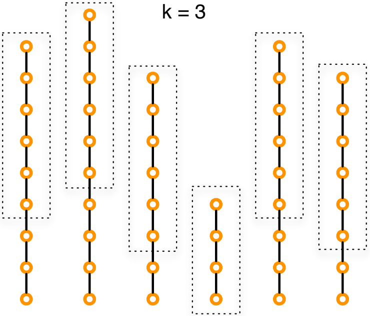

$\rightarrow$ sequence $X_1, \ldots, X_q$ of blocks is a path decomposition of width $\max_{1 \leqslant i \leqslant q} |X_i| - 1 \leqslant 2k \cdot w - 1$

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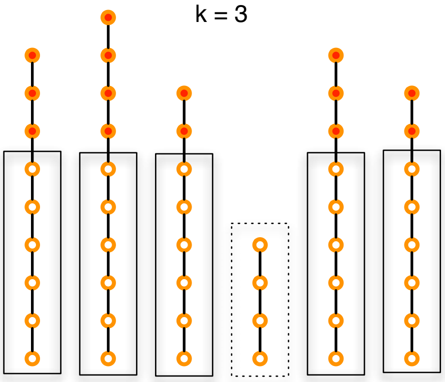

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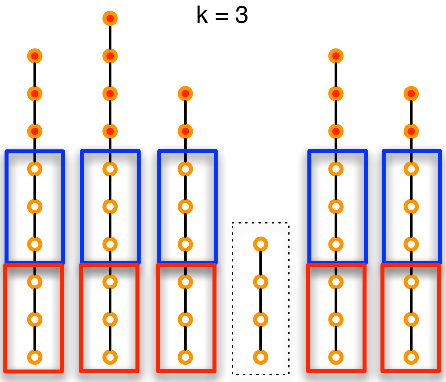

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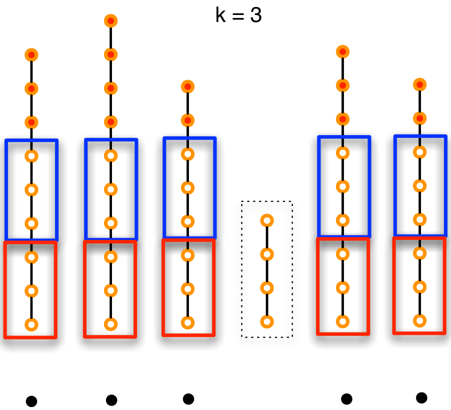

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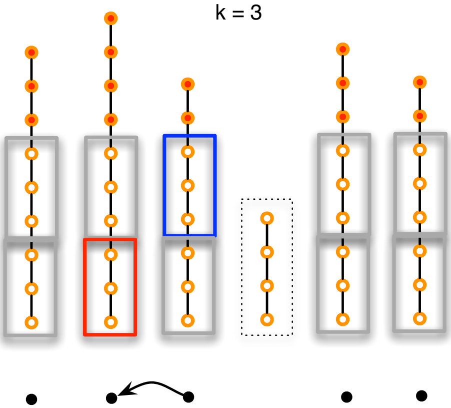

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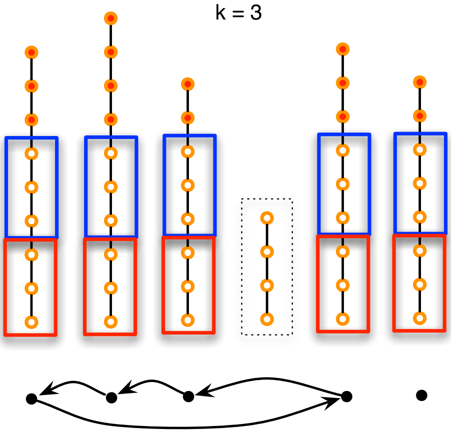

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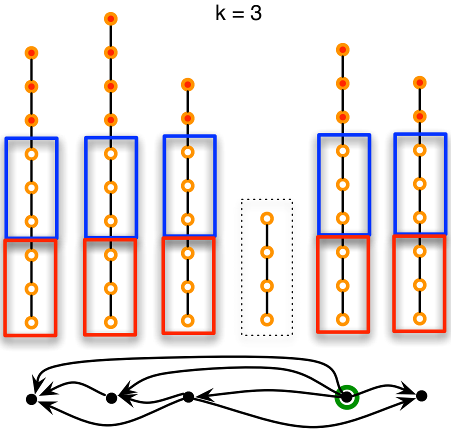

Theorem (DJW)

First-Fit partitions every poset of width w without $\mathbf{k} + \mathbf{k}$ into at most $16kw$ chains

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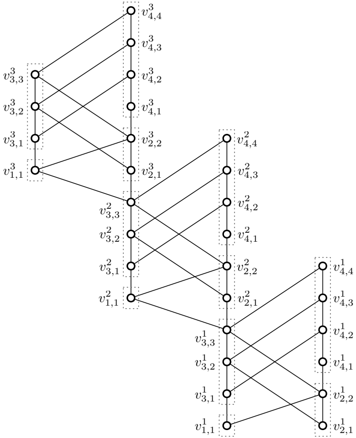
First-Fit can be forced to use $(k-1)(w-1)$ chains:



Thank You!