$\mathcal{Q}_2$-Free Families in the Boolean Lattice

Lucas Kramer,

Joint work with Ryan Martin and Michael Young

Iowa State University

Background

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathcal{Q}_n = (2^{[n]}, \subseteq)$ be a poset of all the subsets of $[n]$ along with the subset relation. This is refered to as the Boolean lattice.

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Let $\mathcal{P}$ be a subposet. A family of subsets $\mathcal{F}$ of $[n]$ is $\mathcal{P}$-free if there is no subposet of $\mathcal{F}$ of the form $\mathcal{P}$. Let $\text{ex}(\mathcal{P})$ to be the maximum sized $\mathcal{P}$-free family.

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In 1928, Sperner [5] proved that the largest family of subsets of $[n]$ for which no one set contains another has size $\binom{n}{\lfloor n/2 \rfloor}$

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Theorem [Erdős, 1945][2]
For $n \geq k-1 \geq 1$, $\mathrm{ex}(n,\mathcal{P}_k) = \sum(n,k-1)$.

Conjecture
For every finite poset $\mathcal{P}$ the limit
$\pi(\mathcal{P}) \stackrel{def}{=} \lim_{n\to\infty} \mathrm{ex}(\mathcal{P})\binom{n}{\lfloor n/2 \rfloor}^{-1}$
exists and is an integer.

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Theorem[Buhk,2009][1]
Let $\mathcal{P}$ be a poset. If $H(\mathcal{P})$ is a tree then
$\mathrm{ex}(\mathcal{P}) = (h(\mathcal{P}) - 1)\binom{n}{\lfloor n/2 \rfloor}(1 + o(1))$.

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Observe that even if we take the $r + 1$ middle layers with an element in the top and bottom layers we have $2^r - 2$ elements in the layers between and hence no $\mathcal{D}_k$ regardless of what n is.

Main Results

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Theorem If $\mathcal{F}$ is a $\mathcal{Q}_2$-free subposet of $\mathcal{Q}_n$ then

$$|\mathcal{F}| \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

Sketch of Proof

If $\mathcal{F}$ is a family of sets in the n-dimensional Boolean lattice, the **Lubell function** of that family is defined to be $\mathrm{Lu}(n,\mathcal{F}) \stackrel{\mathrm{def}}{=} \sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1}$.

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If $\mathcal{F}$ is a family of sets in the n-dimensional Boolean lattice, the **Lubell function** of that family is defined to be $\mathrm{Lu}(n, \mathcal{F}) \stackrel{\mathrm{def}}{=} \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}$.

Let $\mathrm{maxLu}(n, \mathcal{P})$ be the maximum of $\mathrm{Lu}(n, \mathcal{F})$ over all families $\mathcal{F}$ that are both $\mathcal{P}$-free and contain the empty set. Furthermore, set

$$\mathrm{maxLu}(\mathcal{P}) \stackrel{\mathrm{def}}{=} \limsup_{n \to \infty} \{\mathrm{maxLu}(n, \mathcal{P})\}.$$

Sketch of Proof

Lemma 1 Let $\mathcal{Q}_2$ denote the diamond and let $\mathrm{maxLu}(\mathcal{Q}_2)$ be defined as above. Then $|\mathcal{F}| \leq (\mathrm{maxLu}(\mathcal{Q}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$. That is, $\mathrm{ex}(n, \mathcal{Q}_2) \leq (\mathrm{maxLu}(\mathcal{Q}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

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Proven by seperating our set $\mathcal{F}$ into 3 antichains and seperating all the chains into those that hit the minimal elements of $\mathcal{F}$ and those that do not. We make use of the YBML inequality for the chains that miss the minimal elements and then, using a counting arguement, compare the rest to lower order Lubell functions.

Sketch of Proof

For a graph G, let $\alpha_i = \alpha_i(G)$ denote the number of triples that induce exactly i edges for $i = 0, 1, 2, 3$ and let $\beta_j = \beta_j(G)$ induce the number of quadruples that induce exactly j edges for $j = 0, \ldots, 6$. If (X, Y) is an ordered bipartition of $V(G)$, then let $\overline{e}(X)$ denote the number of nonedges in the subgraph induced by X and $\overline{e}(Y)$ denote the number of nonedges in the subgraph induced by Y.

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This allows us to sample and keep track of edges and non-edges which we will need later.

Sketch of Proof

Lemma 2 For every $\mathcal{Q}_2$-free family $\mathcal{F}$ in $\mathcal{Q}_n$ with $\emptyset \in \mathcal{F}$, there exist the following:

- a graph $G = (V, E)$ on $v \leq n$ vertices and
- a set $W = \{w_{v+1}, \ldots, w_n\}$ such that, for each $w \in W$, (X_w, Y_w) is an ordered bipartition of V;

for which $\mathrm{Lu}(n, \mathcal{F}) \leq 2 + f(n, G, W)$, where, with the notation as above,

$$f(n, G, W) = \frac{2\alpha_1(G) - 2\alpha_2(G)}{(n)_3} + \frac{6\beta_0(G)}{(n)_4} + \sum_{w \in W} \left[\frac{|X_w| - |Y_w|}{(n)_2} + \frac{4\overline{e}(Y_w) - 2\overline{e}(X_w)}{(n)_3} \right].$$

Sketch of Proof

The graph G is constructed using the singletons from $\mathcal{Q}_n$ that are not in $\mathcal{F}$ or
$V(G) = \{\{x\} \in \mathcal{Q}_n : \{x\} \notin \mathcal{F}\}$ and the edge set to be the doubletons in $\mathcal{F}$ that have both end points in $V(G)$ or
$E(G) = \{\{x, y\} : \{x, y\} \in \mathcal{F}, x, y \in V(G)\}$.

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By letting Ψ_i be the number of full chains containing exactly i elements of $\mathcal{F}$ we have that
$\mathrm{Lu}(n, \mathcal{F}) = 2 + \frac{|\Psi_3| - |\Psi_1|}{n!}$

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We then proceed by counting full chains that hit various members of $\mathcal{F}$ and use these to place bounds on $|\Psi_1|$ and $|\Psi_3|$.

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In doing this we make several connections to the graph G described before and our bound becomes the graph invariant equation noted in the statement of the Lemma.

Sketch of Proof

Lemma 3 For any integer n, graph $G = (V, E)$ on $v \leq n$ vertices and a set W, of $n - v$ bipartitions of $V(G)$,

$$f(n, G, W) \leq \frac{1}{4} + O\left(\frac{1}{n}\right).$$

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This is the best we can do with the current approach because we use the graph invarient from Lemma 3 which is bounded below by $\frac{1}{4}$.

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Then we have that $\frac{2\alpha_1(G) - 2\alpha_2(G)}{(n)_3} + \frac{6\beta_0(G)}{(n)_4} = \frac{1}{4}$. Hence our maximum value can be at most $\frac{1}{4}$.

Sketch of Proof

To prove this bound is tight we sample the graph G by taking subsets of order 4. We then look at all the densities of these sample graphs within G and then add a set of constants whos some is zero in such a way so that all the densities are less than or equal to $1/4$ which then implies that the value of the function is less than or equal to $1/4$.

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these constants whos sum is nonnegative (zero) to add to our densities was found using Razborov's Flag Algebra method.

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Then all three of the previous lemmas together imply our theorem giving us a value of 2.25.

Further work

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References

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The End

THANK YOU.