Forbidden Induced Posets in the Boolean Lattice

Kevin G. Milans (milans@math.sc.edu)
Joint with L. Lu

University of South Carolina; West Virginia University

SIAM Conference on Discrete Mathematics
Dalhousie University
Halifax, Nova Scotia, Canada
18 June 2012

Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$

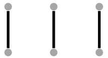
Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$
- P is an induced subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \iff f(x) \leq_Q f(y).$$

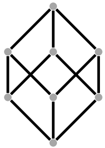
Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$
- P is an induced subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \iff f(x) \leq_Q f(y).$$

Example



$3 \cdot \mathcal{P}_2$

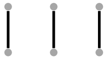


$2^{[3]}$

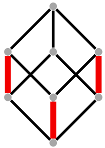
Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$
- P is an induced subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \iff f(x) \leq_Q f(y).$$

Example



$3 \cdot \mathcal{P}_2$



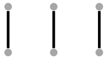
$2^{[3]}$

- $3 \cdot \mathcal{P}_2$ is a subposet of $2^{[3]}$, but

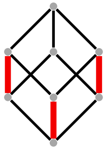
Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$
- P is an induced subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \iff f(x) \leq_Q f(y).$$

Example



$3 \cdot \mathcal{P}_2$



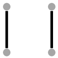
$2^{[3]}$

- $3 \cdot \mathcal{P}_2$ is a subposet of $2^{[3]}$, but
- not an induced subset.

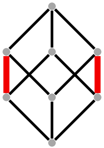
Poset Containment

- P is a subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \implies f(x) \leq_Q f(y).$$
- P is an induced subposet of Q if there is an injection $f\colon P \to Q$ such that
$$\forall x, y \in P \quad x \leq_P y \iff f(x) \leq_Q f(y).$$

Example



$2 \cdot \mathcal{P}_2$



$2^{[3]}$

- $3 \cdot \mathcal{P}_2$ is a subposet of $2^{[3]}$, but
- not an induced subset.
- $2 \cdot \mathcal{P}_2$ is an induced subposet of $2^{[3]}$.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.
- Let 2^A be the inclusion order on subsets of A.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.
- Let 2^A be the inclusion order on subsets of A.
- The *n*-dimensional Boolean lattice is $2^{[n]}$.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.
- Let 2^A be the inclusion order on subsets of A.
- The *n-dimensional Boolean lattice* is $2^{[n]}$.
- When $\mathcal{F} \subseteq 2^{[n]}$, we view $\mathcal{F}$ as a poset ordered by inclusion.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.
- Let 2^A be the inclusion order on subsets of A.
- The n-dimensional Boolean lattice is $2^{[n]}$.
- When $\mathcal{F} \subseteq 2^{[n]}$, we view $\mathcal{F}$ as a poset ordered by inclusion.
- Let $\mathrm{La}(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not a subposet of $\mathcal{F}$.

The Turán Problem

- Let $[n] = \{1, \ldots, n\}$.
- Let 2^A be the inclusion order on subsets of A.
- The n-dimensional Boolean lattice is $2^{[n]}$.
- When $\mathcal{F} \subseteq 2^{[n]}$, we view $\mathcal{F}$ as a poset ordered by inclusion.
- Let $\mathrm{La}(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not a subposet of $\mathcal{F}$.

Theorem (Sperner (1928); Erdős (1945))

$\mathrm{La}(n, \mathcal{P}_k)$ *equals the sum of the* $k-1$ *largest binomial coefficients in* $\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\}$. *For fixed* k *and* $n \to \infty$,

$$\mathrm{La}(n, \mathcal{P}_k) = (k - 1 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

The Turán Threshold

- The Turán threshold of P, denoted $\pi(P)$, is given by

$$\pi(P) = \limsup_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor n/2\rfloor}}.$$

The Turán Threshold

- The Turán threshold of P, denoted $\pi(P)$, is given by

$$\pi(P) = \limsup_{n\to\infty} \frac{\mathrm{La}(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- [Sperner (1928); Erdős (1945)] $\pi(\mathcal{P}_k) = k - 1$

The Turán Threshold

- The Turán threshold of P, denoted $\pi(P)$, is given by

$$\pi(P) = \limsup_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- [Sperner (1928); Erdős (1945)] $\pi(\mathcal{P}_k) = k - 1$
- Always $\pi(P) \leq |P| - 1$.

The Turán Threshold

- The Turán threshold of P, denoted $\pi(P)$, is given by

$$\pi(P) = \limsup_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor n/2\rfloor}}.$$

- [Sperner (1928); Erdős (1945)] $\pi(\mathcal{P}_k) = k-1$
- Always $\pi(P) \leq |P| - 1$.
- The height of P is the size of a longest chain in P.

The Turán Threshold

- The Turán threshold of P, denoted $\pi(P)$, is given by

$$\pi(P) = \limsup_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- [Sperner (1928); Erdős (1945)] $\pi(\mathcal{P}_k) = k - 1$
- Always $\pi(P) \le |P| - 1$.
- The height of P is the size of a longest chain in P.

Theorem (Bukh (2010))

If the Hasse diagram of P is a tree, then $\pi(P) = h(P) - 1$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n\to\infty} \frac{\mathrm{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n\to\infty} \frac{\mathrm{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- Fact: $\mathrm{La}(n, P) \leq \mathrm{La}^*(n, P)$ and $\pi(P) \leq \pi^*(P)$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n,P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n\to\infty} \frac{\mathrm{La}^*(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- Fact: $\mathrm{La}(n,P) \leq \mathrm{La}^*(n,P)$ and $\pi(P) \leq \pi^*(P)$.
- [Sperner (1928); Erdős (1945)] $\pi^*(\mathcal{P}_k) = k - 1$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n\to\infty} \frac{\mathrm{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- Fact: $\mathrm{La}(n, P) \leq \mathrm{La}^*(n, P)$ and $\pi(P) \leq \pi^*(P)$.
- [Sperner (1928); Erdős (1945)] $\pi^*(\mathcal{P}_k) = k - 1$.
- [Carroll–Katona (2008)]: sharp bounds on $\mathrm{La}^*(n,$  $)$, implying $\pi^*($  $) = 1$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n \to \infty} \frac{\mathrm{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- Fact: $\mathrm{La}(n, P) \leq \mathrm{La}^*(n, P)$ and $\pi(P) \leq \pi^*(P)$.
- [Sperner (1928); Erdős (1945)] $\pi^*(\mathcal{P}_k) = k - 1$.
- [Carroll–Katona (2008)]: sharp bounds on $\mathrm{La}^*(n,$ $)$, implying $\pi^*($$) = 1$.

Theorem (Boehnlein–Jiang (2011))

If the Hasse diagram of P is a tree, then $\pi^*(P) = h(P) - 1$.

The Induced Turán Threshold

- Let $\mathrm{La}^*(n, P)$ be the maximum size of a family $\mathcal{F}$ such that $\mathcal{F} \subseteq 2^{[n]}$ and P is not an *induced* subposet of $\mathcal{F}$.
- The induced Turán threshold of P, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n\to\infty} \frac{\mathrm{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$

- Fact: $\mathrm{La}(n, P) \le \mathrm{La}^*(n, P)$ and $\pi(P) \le \pi^*(P)$.
- [Sperner (1928); Erdős (1945)] $\pi^*(\mathcal{P}_k) = k - 1$.
- [Carroll–Katona (2008)]: sharp bounds on $\mathrm{La}^*(n,$  $)$, implying $\pi^*($  $) = 1$.

Theorem (Boehnlein–Jiang (2011))

If the Hasse diagram of P is a tree, then $\pi^*(P) = h(P) - 1$.

Observation (Boehnlein–Jiang (2011))

$\pi^*(P)$ may be much larger than $\pi(P)$.

Our Results

Theorem

If P is a series-parallel poset, then $\pi^*(P)$ is finite.

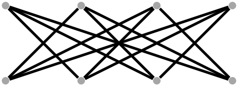
Our Results

Theorem

If P is a series-parallel poset, then $\pi^*(P)$ is finite.

Definition

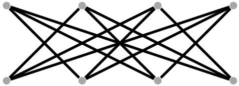
The standard example on $2r$ elements, denoted $\mathcal{S}_r$, is the poset consisting of antichains $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ and the relations $a_i \leq b_j$ where $i \neq j$.



Our Results

Theorem
If P is a series-parallel poset, then $\pi^*(P)$ is finite.

Definition
The standard example on $2r$ elements, denoted $\mathcal{S}_r$, is the poset consisting of antichains $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ and the relations $a_i \leq b_j$ where $i \neq j$.



Theorem
$r - 2 \leq \pi^*(\mathcal{S}_r) \leq 4r + O(\sqrt{r})$

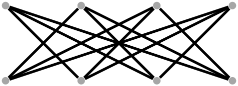
Our Results

Theorem

If P is a series-parallel poset, then $\pi^*(P)$ is finite.

Definition

The standard example on $2r$ elements, denoted $\mathcal{S}_r$, is the poset consisting of antichains $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ and the relations $a_i \leq b_j$ where $i \neq j$.



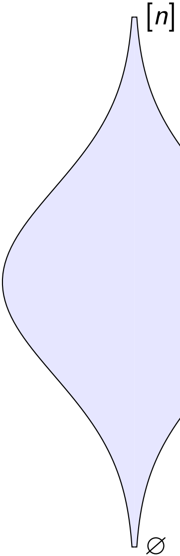
Theorem

$r - 2 \leq \pi^*(\mathcal{S}_r) \leq 4r + O(\sqrt{r})$

Corollary

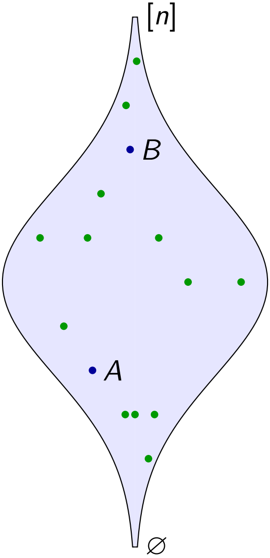
$\pi^*(2^{[3]}) \leq 23.55.$

The Lubell Function



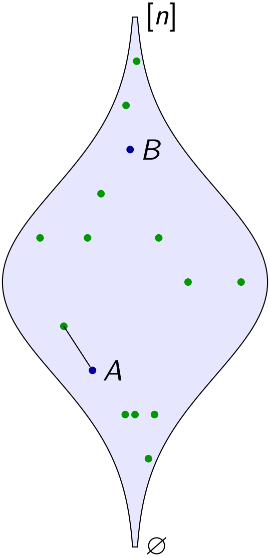
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



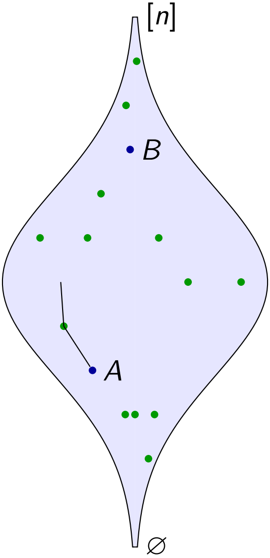
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



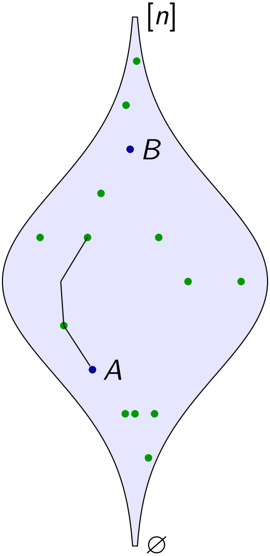
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



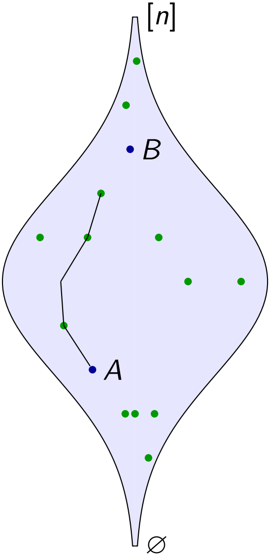
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F};[A,B])$ to be the expected number of times that a random full (A,B)-chain meets $\mathcal{F}$.

The Lubell Function



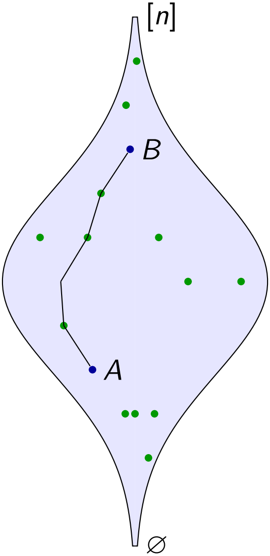
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



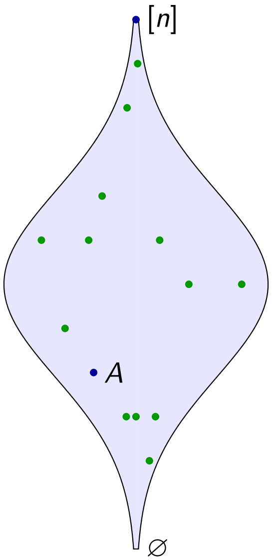
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



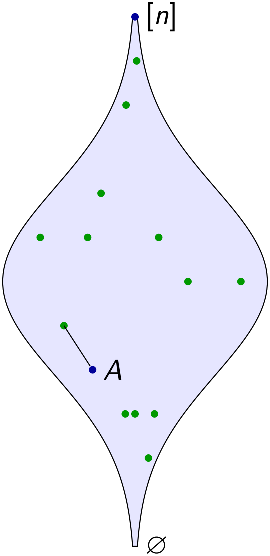
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.

The Lubell Function



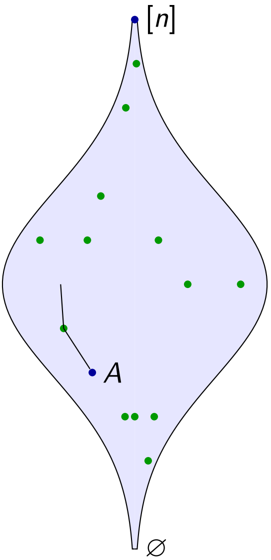
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



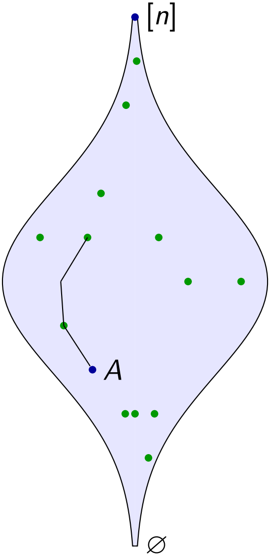
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



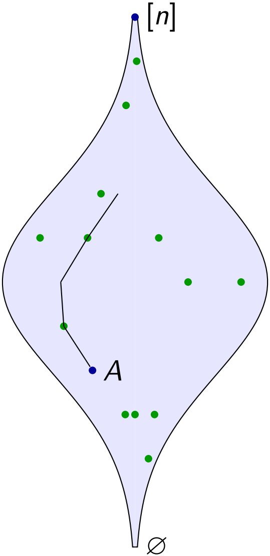
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



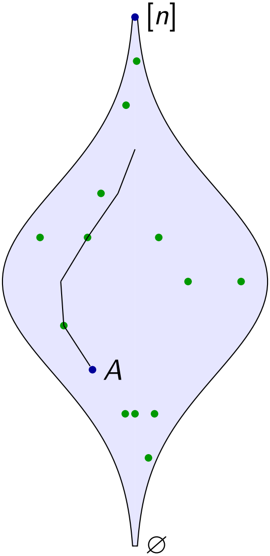
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



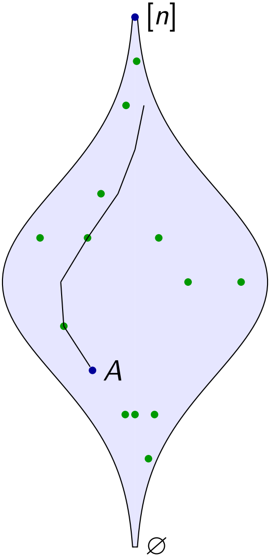
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell^+_A(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



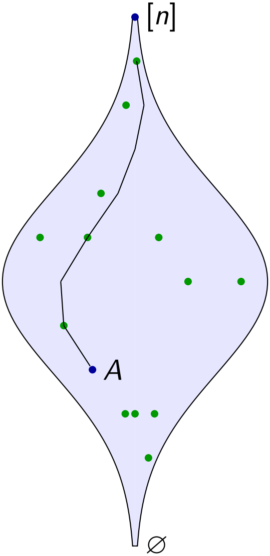
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



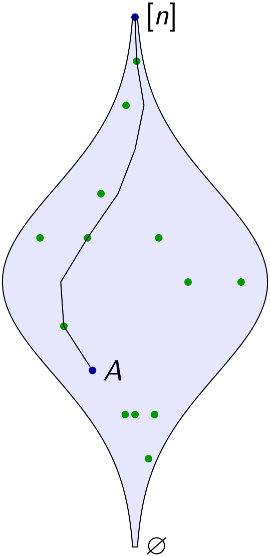
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



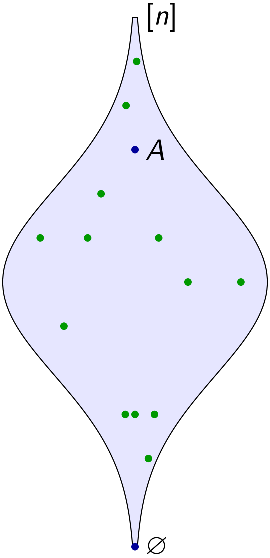
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



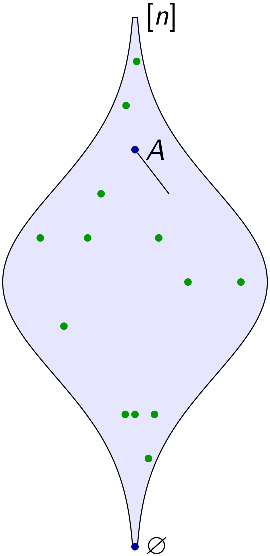
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.

The Lubell Function



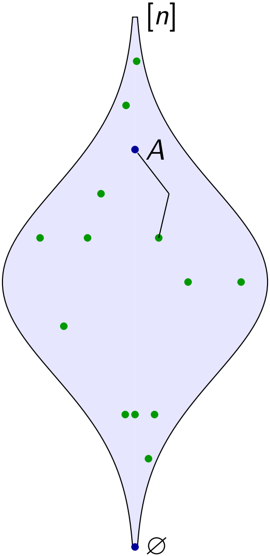
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell^+_A(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell^-_A(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



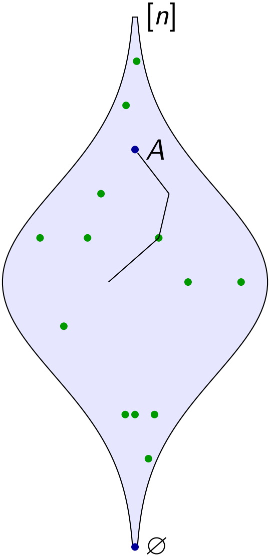
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



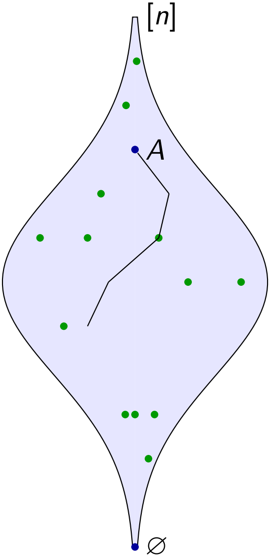
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



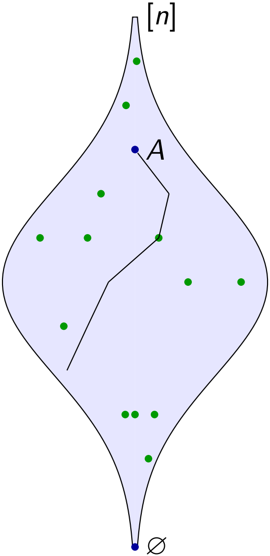
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



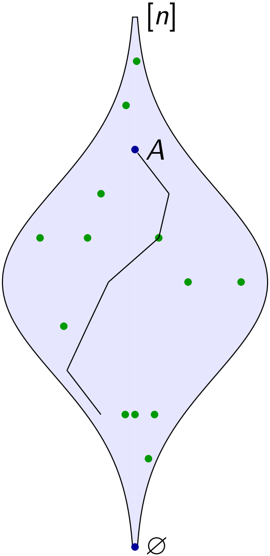
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



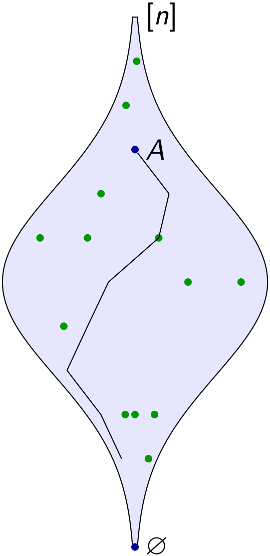
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



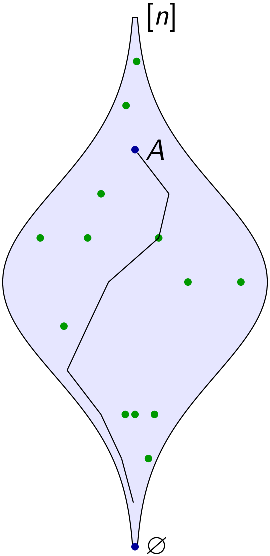
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



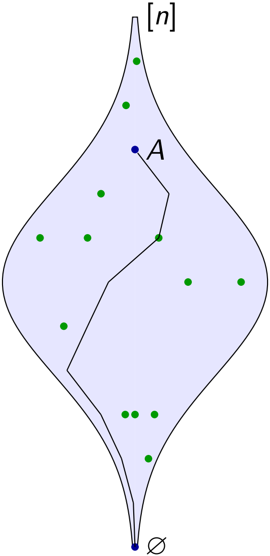
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell^+_A(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell^-_A(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



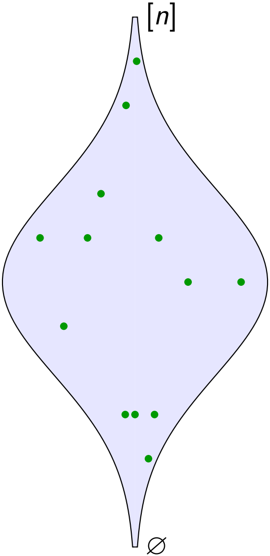
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



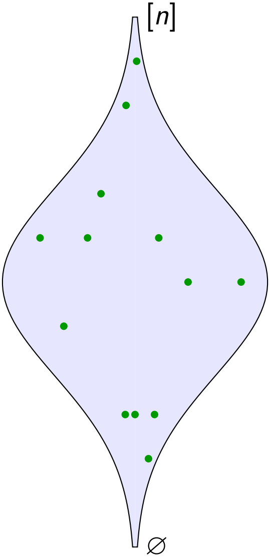
- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.

The Lubell Function



- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.
- $\ell(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, [n]])$.

The Lubell Function



- Given $\mathcal{F} \subseteq 2^{[n]}$ and $A \subseteq B$, define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times that a random full (A, B)-chain meets $\mathcal{F}$.
- $\ell_A^+(\mathcal{F}) = \ell(\mathcal{F}; [A, [n]])$.
- $\ell_A^-(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, A])$.
- $\ell(\mathcal{F}) = \ell(\mathcal{F}; [\varnothing, [n]])$.
- Linearity of Expectation:

$$\ell(\mathcal{F}) = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}$$

The Induced Lubell Threshold

- The induced Lubell threshold, denoted $\lambda^*(P)$, is given by

  $\lambda^*(P) = \sup\{\ell(\mathcal{F})\colon \mathcal{F}$ does not contain an induced copy of $P\}$.

The Induced Lubell Threshold

- The induced Lubell threshold, denoted $\lambda^*(P)$, is given by

  $$\lambda^*(P) = \sup\{\ell(\mathcal{F})\colon \mathcal{F} \text{ does not contain an induced copy of } P\}.$$

- $\ell(\mathcal{F}) > \lambda^*(P) \implies \mathcal{F}$ contains an induced copy of P.

The Induced Lubell Threshold

- The induced Lubell threshold, denoted $\lambda^*(P)$, is given by

  $\lambda^*(P) = \sup\{\ell(\mathcal{F})\colon\ \mathcal{F}$ does not contain an induced copy of $P\}$.

- $\ell(\mathcal{F}) > \lambda^*(P) \implies \mathcal{F}$ contains an induced copy of P.
- $\pi^*(P) \leq \lambda^*(P)$.

The Induced Lubell Threshold

- The induced Lubell threshold, denoted $\lambda^*(P)$, is given by

  $$\lambda^*(P) = \sup\{\ell(\mathcal{F})\colon \mathcal{F} \text{ does not contain an induced copy of } P\}.$$

- $\ell(\mathcal{F}) > \lambda^*(P) \implies \mathcal{F}$ contains an induced copy of P.
- $\pi^*(P) \leq \lambda^*(P)$.

Conjecture

Always $\lambda^*(P)$ is finite.

Series-Parallel Posets

Theorem

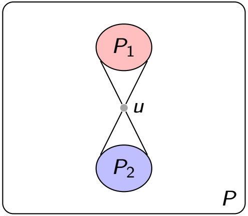
If P is a series-parallel poset, then $\lambda^*(P)$ is finite.

Series-Parallel Posets

Theorem

If P is a series-parallel poset, then $\lambda^*(P)$ is finite.

Lemma (Series Construction)



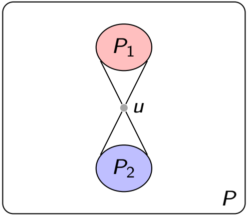
$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Series-Parallel Posets

Theorem

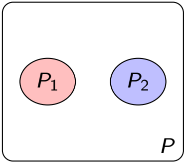
If P is a series-parallel poset, then $\lambda^*(P)$ is finite.

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Lemma (Parallel Construction)



$$\lambda^*(P) \leq \max\{\lambda^*(P_1), \lambda^*(P_2)\} + 8$$

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell^+_A(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the # of times a random full chain meets $\mathcal{F}$.

□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the # of times a random full chain meets $\mathcal{F}$.
- When $X \geq 1$, let Y be the first element in $\mathcal{F}$.

□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell^+_A(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the # of times a random full chain meets $\mathcal{F}$.
- When $X \geq 1$, let Y be the first element in $\mathcal{F}$.

$$\ell(\mathcal{F}) = \mathbf{E}[X] = \sum_{A \in \mathcal{F}} \mathbf{E}[X | Y = A] \cdot \Pr[Y = A]$$

□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the # of times a random full chain meets $\mathcal{F}$.
- When $X \geq 1$, let Y be the first element in $\mathcal{F}$.

$$\begin{aligned}
\ell(\mathcal{F}) = \mathbf{E}[X] &= \sum_{A \in \mathcal{F}} \mathbf{E}[X | Y = A] \cdot \Pr[Y = A] \\
&= \sum_{A \in \mathcal{F}} \ell_A^+(\mathcal{F}) \cdot \Pr[Y = A]
\end{aligned}$$

□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the $\#$ of times a random full chain meets $\mathcal{F}$.
- When $X \geq 1$, let Y be the first element in $\mathcal{F}$.

$$\begin{aligned}
\ell(\mathcal{F}) = \mathbf{E}[X] &= \sum_{A \in \mathcal{F}} \mathbf{E}[X \mid Y = A] \cdot \Pr[Y = A] \\
&= \sum_{A \in \mathcal{F}} \ell_A^+(\mathcal{F}) \cdot \Pr[Y = A] \\
&\leq \sum_{A \in \mathcal{F}} \alpha \cdot \Pr[Y = A]
\end{aligned}$$

□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Proof.

- Let X be the # of times a random full chain meets $\mathcal{F}$.
- When $X \geq 1$, let Y be the first element in $\mathcal{F}$.

$$\begin{aligned}
\ell(\mathcal{F}) = \mathbf{E}[X] &= \sum_{A \in \mathcal{F}} \mathbf{E}[X | Y = A] \cdot \Pr[Y = A] \\
&= \sum_{A \in \mathcal{F}} \ell_A^+(\mathcal{F}) \cdot \Pr[Y = A] \\
&\leq \sum_{A \in \mathcal{F}} \alpha \cdot \Pr[Y = A] \\
&\leq \alpha
\end{aligned}$$

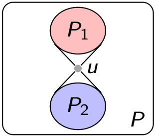
□

Series Construction

Lemma (Shallow Sets)

A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Lemma (Series Construction)



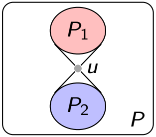
$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Series Construction

Lemma (Shallow Sets)

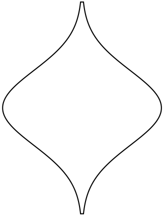
A set $A \in \mathcal{F}$ is *α-shallow* *if $\ell^+_A(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.*

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- Let $\alpha_1 = \lambda^*(P_1)$ and $\alpha_2 = \lambda^*(P_2)$.

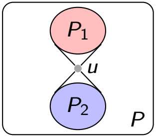
□

Series Construction

Lemma (Shallow Sets)

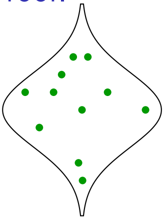
A set $A \in \mathcal{F}$ is α*-shallow* *if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.*

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- Let $\alpha_1 = \lambda^*(P_1)$ and $\alpha_2 = \lambda^*(P_2)$.
- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > \alpha_1 + \alpha_2 + 2$.

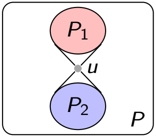
□

Series Construction

Lemma (Shallow Sets)

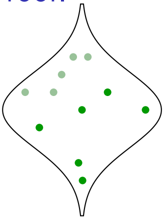
A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- Let $\alpha_1 = \lambda^*(P_1)$ and $\alpha_2 = \lambda^*(P_2)$.
- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > \alpha_1 + \alpha_2 + 2$.
- Discard $(\alpha_1 + 1)$-shallow points.

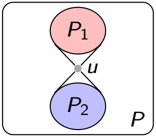
□

Series Construction

Lemma (Shallow Sets)

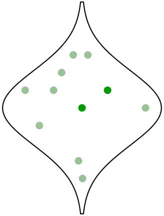
A set $A \in \mathcal{F}$ is α*-shallow* *if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.*

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- Let $\alpha_1 = \lambda^*(P_1)$ and $\alpha_2 = \lambda^*(P_2)$.
- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > \alpha_1 + \alpha_2 + 2$.
- Discard $(\alpha_1 + 1)$-shallow points.
- Discard "dually" $(\alpha_2 + 1)$-shallow points.

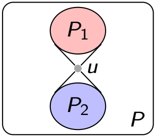
□

Series Construction

Lemma (Shallow Sets)

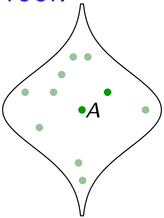
A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- Let $\alpha_1 = \lambda^*(P_1)$ and $\alpha_2 = \lambda^*(P_2)$.
- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > \alpha_1 + \alpha_2 + 2$.
- Discard $(\alpha_1 + 1)$-shallow points.
- Discard "dually" $(\alpha_2 + 1)$-shallow points.
- Choose a surviving set A.

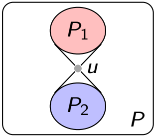
□

Series Construction

Lemma (Shallow Sets)

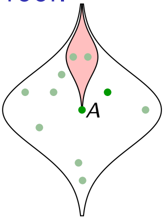
A set $A \in \mathcal{F}$ is *α-shallow* *if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.*

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- $\ell_A^+(\mathcal{F}) > \alpha_1 + 1$

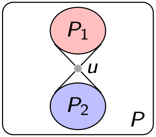
□

Series Construction

Lemma (Shallow Sets)

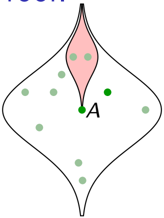
A set $A \in \mathcal{F}$ is *α-shallow* *if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.*

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- $\ell_A^+(\mathcal{F}) > \alpha_1 + 1$
- Find P_1 in $(\mathcal{F} - \{A\}) \cap [A, [n]]$.

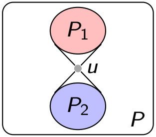
□

Series Construction

Lemma (Shallow Sets)

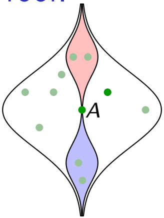
A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- $\ell_A^+(\mathcal{F}) > \alpha_1 + 1$
- Find P_1 in $(\mathcal{F} - \{A\}) \cap [A, [n]]$.
- $\ell_A^-(\mathcal{F}) > \alpha_2 + 1$

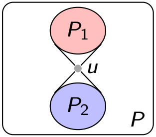
□

Series Construction

Lemma (Shallow Sets)

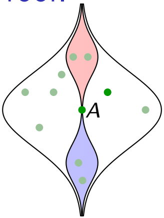
A set $A \in \mathcal{F}$ is α-shallow if $\ell_A^+(\mathcal{F}) \leq \alpha$. If every element in $\mathcal{F}$ is α-shallow, then $\ell(\mathcal{F}) \leq \alpha$.

Lemma (Series Construction)



$$\lambda^*(P) \leq \lambda^*(P_1) + \lambda^*(P_2) + 2$$

Proof.



- $\ell_A^+(\mathcal{F}) > \alpha_1 + 1$
- Find P_1 in $(\mathcal{F} - \{A\}) \cap [A, [n]]$.
- $\ell_A^-(\mathcal{F}) > \alpha_2 + 1$
- Find P_2 in $(\mathcal{F} - \{A\}) \cap [\varnothing, A)$.

□

Parallel Construction

- $\mathcal{F}$ is balanced if, for each $A, B \in 2^{[n]}$ with $A \subseteq B$, we have $\ell(\mathcal{F}; [A, B]) \leq \ell(\mathcal{F})$.

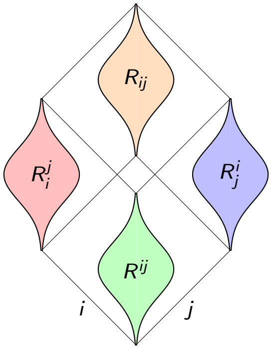
Parallel Construction

- $\mathcal{F}$ is balanced if, for each $A, B \in 2^{[n]}$ with $A \subseteq B$, we have $\ell(\mathcal{F}; [A, B]) \leq \ell(\mathcal{F})$.
- Given disjoint sets A and B, let $R_A^B = [A, [n] - B]$; this is the sublattice consisting of all sets that contain A and are disjoint from B.

Parallel Construction

- $\mathcal{F}$ is balanced if, for each $A, B \in 2^{[n]}$ with $A \subseteq B$, we have $\ell(\mathcal{F}; [A, B]) \leq \ell(\mathcal{F})$.
- Given disjoint sets A and B, let $R_A^B = [A, [n] - B]$; this is the sublattice consisting of all sets that contain A and are disjoint from B.

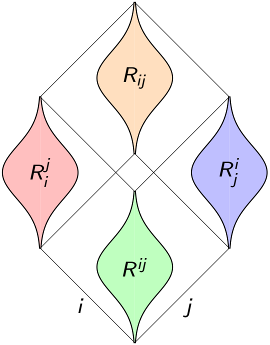
Lemma (Four Sublattices)



If $\mathcal{F}$ is balanced, then there exist $i, j \in [n]$ such that the Lubell Function in each of the sublattices $R^{ij}, R_i^j, R_j^i, R_{ij}$ is at least $\ell(\mathcal{F}) - 8$.

Parallel Construction

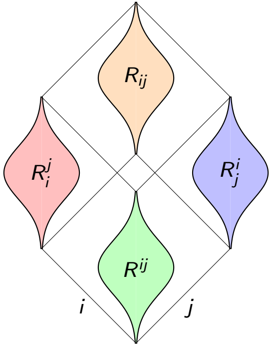
Lemma (Four Sublattices)



If $\mathcal{F}$ is balanced, then there exist $i, j \in [n]$ such that the Lubell Function in each of the sublattices $R^{ij}, R_i^j, R_j^i, R_{ij}$ is at least $\ell(\mathcal{F}) - 8$.

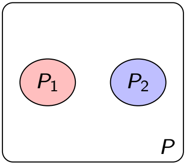
Parallel Construction

Lemma (Four Sublattices)



If $\mathcal{F}$ is balanced, then there exist $i, j \in [n]$ such that the Lubell Function in each of the sublattices $R^{ij}, R_i^j, R_j^i, R_{ij}$ is at least $\ell(\mathcal{F}) - 8$.

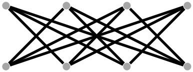
Lemma (Parallel Construction)



$$\lambda^*(P) \leq \max\{\lambda^*(P_1), \lambda^*(P_2)\} + 8$$

The Standard Example and Flexible Sets

Theorem

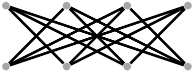


$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

The Standard Example and Flexible Sets

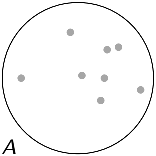
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Definition

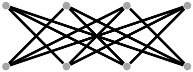


A

We say that $A \in \mathcal{F}$ is γ-flexible if

The Standard Example and Flexible Sets

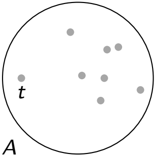
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

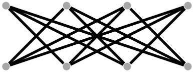
Definition



We say that $A \in \mathcal{F}$ is γ-flexible if there are more than $\gamma|A|$ indices $t \in A$

The Standard Example and Flexible Sets

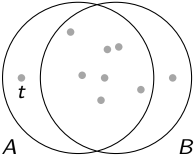
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

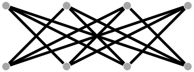
Definition



We say that $A \in \mathcal{F}$ is γ-flexible if there are more than $\gamma|A|$ indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}$ with $|B| = |A|$.

The Standard Example and Flexible Sets

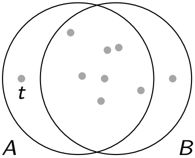
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Definition



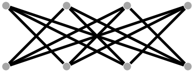
We say that $A \in \mathcal{F}$ is γ-flexible if there are more than $\gamma|A|$ indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}$ with $|B| = |A|$.

Lemma (Flexible Sets)

Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $|A| \leq n/2$ for each $A \in \mathcal{F}$ and suppose that $0 \leq \gamma < 1$.

The Standard Example and Flexible Sets

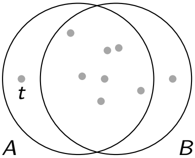
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Definition

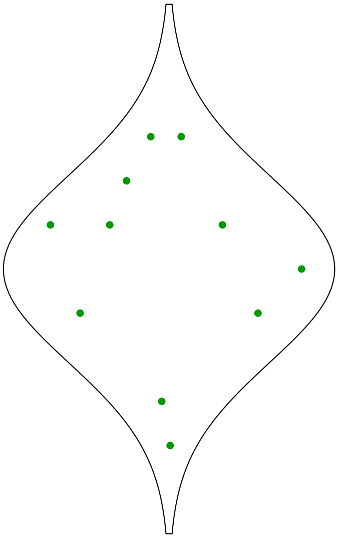


We say that $A \in \mathcal{F}$ is γ-flexible if there are more than $\gamma|A|$ indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}$ with $|B| = |A|$.

Lemma (Flexible Sets)

Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $|A| \leq n/2$ for each $A \in \mathcal{F}$ and suppose that $0 \leq \gamma < 1$. If $\mathcal{F}$ does not contain a γ-flexible set, then $\ell(\mathcal{F}) \leq 1 + \ln 2/(1 - \gamma)$.

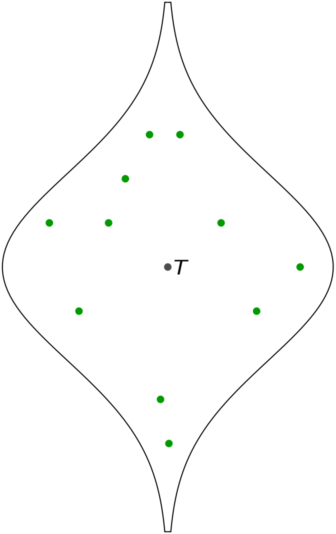
Projection Lemma



Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*

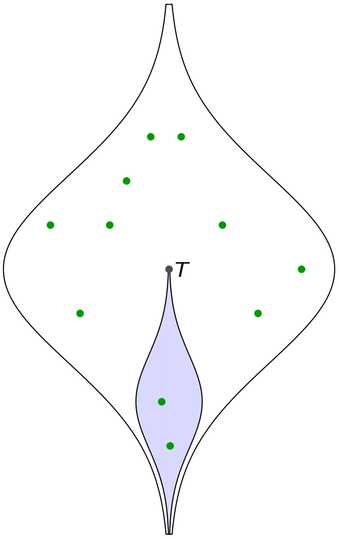
Projection Lemma



Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*
- *Let $T \subseteq [n]$ and let $t = |T|$.*

Projection Lemma

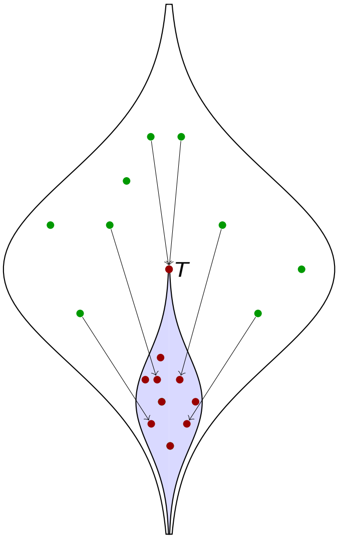


Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*
- *Let $T \subseteq [n]$ and let $t = |T|$.*
- *Let $\mathcal{F}'$ be the projection of $\mathcal{F}$ onto T:*

$$\mathcal{F}' = \{A \cap T : A \in \mathcal{F}\}$$

Projection Lemma

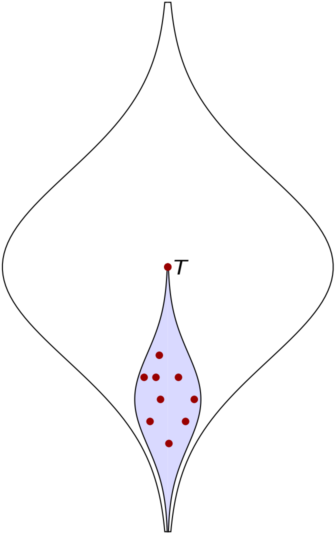


Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*
- *Let $T \subseteq [n]$ and let $t = |T|$.*
- *Let $\mathcal{F}'$ be the projection of $\mathcal{F}$ onto T:*

$$\mathcal{F}' = \{A \cap T \colon A \in \mathcal{F}\}$$

Projection Lemma

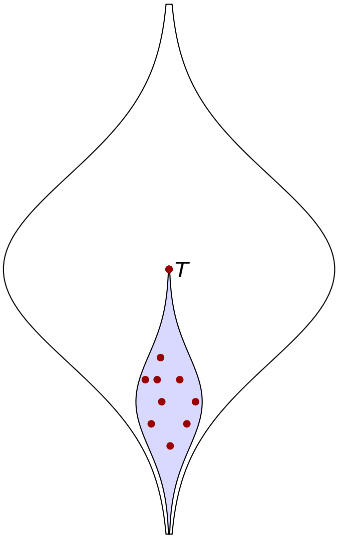


Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*
- *Let $T \subseteq [n]$ and let $t = |T|$.*
- *Let $\mathcal{F}'$ be the projection of $\mathcal{F}$ onto T:*

$$\mathcal{F}' = \{A \cap T : A \in \mathcal{F}\}$$

Projection Lemma



Lemma (Projection)

- *Let $\mathcal{F}$ be a family of sets in $2^{[n]}$.*
- *Let $T \subseteq [n]$ and let $t = |T|$.*
- *Let $\mathcal{F}'$ be the projection of $\mathcal{F}$ onto T:*

$$\mathcal{F}' = \{A \cap T : \ A \in \mathcal{F}\}$$

We have that

$$\ell_T^-(\mathcal{F}') \geq \frac{t+1}{n+1} \cdot \ell(\mathcal{F}).$$

Private Elements Lemma

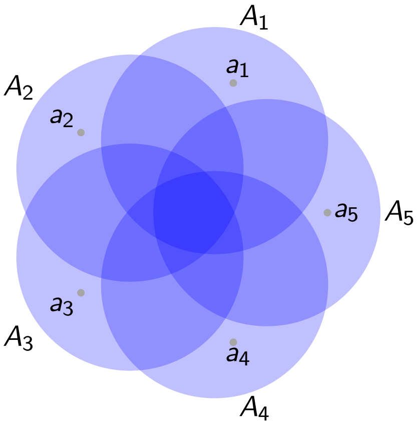
Lemma (Private Elements)

Let r be an integer and suppose that $\mathcal{F} \subseteq 2^{[n]}$ with $\ell(\mathcal{F}) > 2r - 1$.

Private Elements Lemma

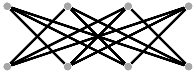
Lemma (Private Elements)

Let r be an integer and suppose that $\mathcal{F} \subseteq 2^{[n]}$ with $\ell(\mathcal{F}) > 2r - 1$. There exist $A_1, \ldots, A_r \in \mathcal{F}$ and $a_1, \ldots, a_r \in [n]$ such that $a_i \in A_j$ if and only if $i \neq j$.



The Standard Example

Theorem

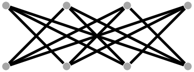


$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

The Standard Example

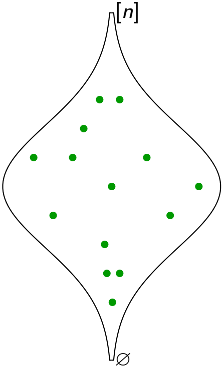
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

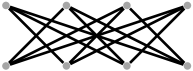


- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > 4r + \Omega(\sqrt{r})$.

□

The Standard Example

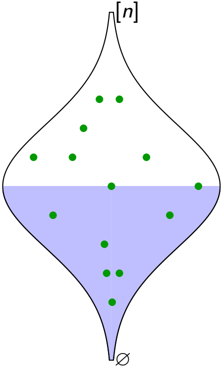
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

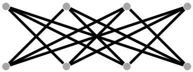


- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > 4r + \Omega(\sqrt{r})$.
- Let $\mathcal{F}_1 = \{A \in \mathcal{F} \colon |A| \leq n/2\}$.

□

The Standard Example

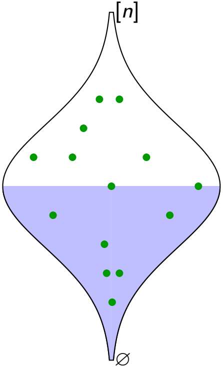
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

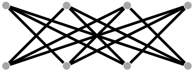


- Consider $\mathcal{F}$ with $\ell(\mathcal{F}) > 4r + \Omega(\sqrt{r})$.
- Let $\mathcal{F}_1 = \{A \in \mathcal{F} \colon |A| \leq n/2\}$.
- By symmetry, we may assume that $\ell(\mathcal{F}_1) > 2r + \Omega(\sqrt{r})$.

□

The Standard Example

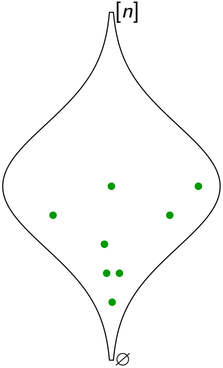
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

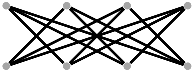


- Set $\gamma = 1 - \frac{c}{\sqrt{r}}$.

□

The Standard Example

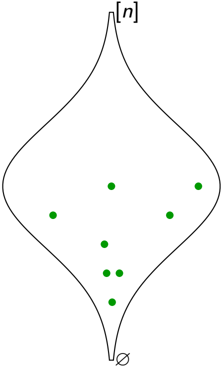
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

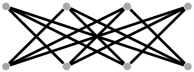


- Set $\gamma = 1 - \frac{c}{\sqrt{r}}$.
- Discard the set $\mathcal{B}_1$ of all points in $\mathcal{F}_1$ that are not γ-flexible. Set $\mathcal{F}_2 = \mathcal{F}_1 - \mathcal{B}_1$.

□

The Standard Example

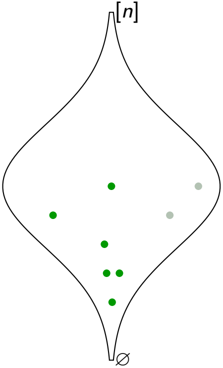
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

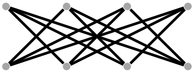


- Set $\gamma = 1 - \frac{c}{\sqrt{r}}$.
- Discard the set $\mathcal{B}_1$ of all points in $\mathcal{F}_1$ that are not γ-flexible. Set $\mathcal{F}_2 = \mathcal{F}_1 - \mathcal{B}_1$.

□

The Standard Example

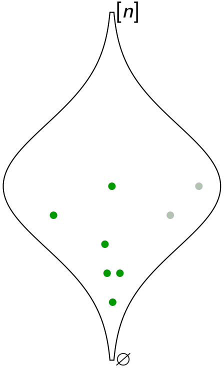
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

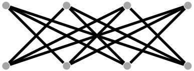


- Set $\gamma = 1 - \frac{c}{\sqrt{r}}$.
- Discard the set $\mathcal{B}_1$ of all points in $\mathcal{F}_1$ that are not γ-flexible. Set $\mathcal{F}_2 = \mathcal{F}_1 - \mathcal{B}_1$.
- By the Flexible Sets Lemma, $\ell(\mathcal{B}_1) < O(\sqrt{r})$.

□

The Standard Example

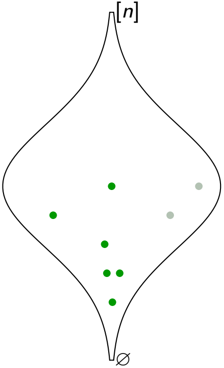
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

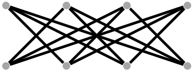


- Set $\gamma = 1 - \frac{c}{\sqrt{r}}$.
- Discard the set $\mathcal{B}_1$ of all points in $\mathcal{F}_1$ that are not γ-flexible. Set $\mathcal{F}_2 = \mathcal{F}_1 - \mathcal{B}_1$.
- By the Flexible Sets Lemma, $\ell(\mathcal{B}_1) < O(\sqrt{r})$.
- We have $\ell(\mathcal{F}_2) > (2r-1)/\gamma$.

□

The Standard Example

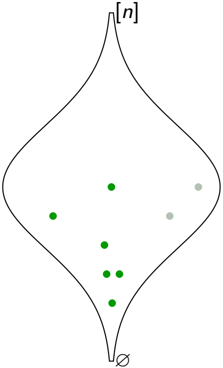
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

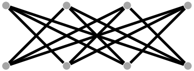


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.

□

The Standard Example

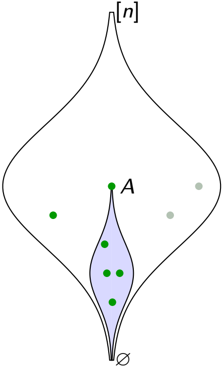
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

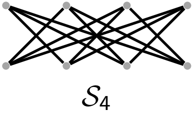


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.

□

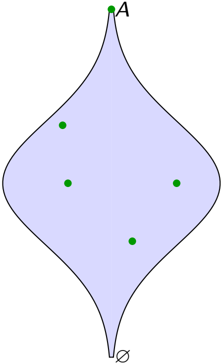
The Standard Example

Theorem



$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

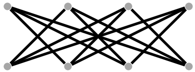


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.

□

The Standard Example

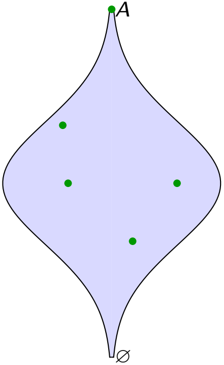
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

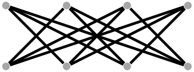


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.
- Let $\mathcal{F}_3 = \{A' \in \mathcal{F}_2 \colon A' \subseteq A\}$.

□

The Standard Example

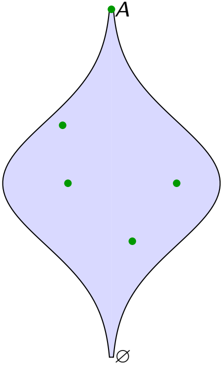
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

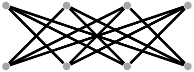


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.
- Let $\mathcal{F}_3 = \{A' \in \mathcal{F}_2 \colon\ A' \subseteq A\}$.
- Let T be the set of all indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}_1$ with $|B| = |A|$.

□

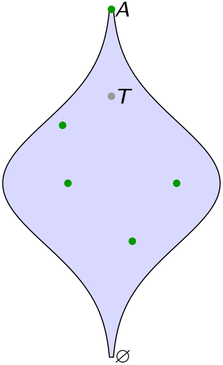
The Standard Example

Theorem



$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

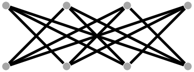


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell^-_A(\mathcal{F}_2) > (2r-1)/\gamma$.
- Let $\mathcal{F}_3 = \{A' \in \mathcal{F}_2\colon\ A' \subseteq A\}$.
- Let T be the set of all indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}_1$ with $|B| = |A|$.

□

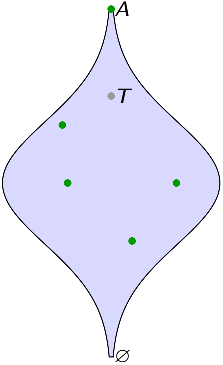
The Standard Example

Theorem



$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

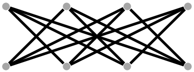


- By the Shallow Sets Lemma, find $A \in \mathcal{F}_2$ with $\ell_A^-(\mathcal{F}_2) > (2r-1)/\gamma$.
- Let $\mathcal{F}_3 = \{A' \in \mathcal{F}_2 \colon A' \subseteq A\}$.
- Let T be the set of all indices $t \in A$ such that $A - B = \{t\}$ for some $B \in \mathcal{F}_1$ with $|B| = |A|$.
- Since A is γ-flexible, $|T| \geq \gamma |A|$.

□

The Standard Example

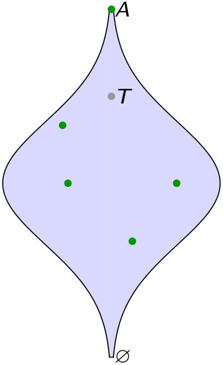
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \le 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

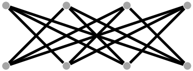


- Let $\mathcal{F}_4$ be the projection of $\mathcal{F}_3$ onto T.

□

The Standard Example

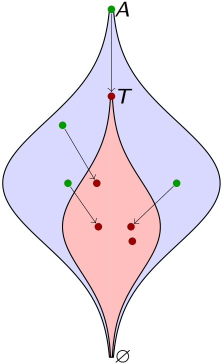
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

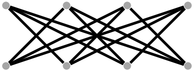


- Let $\mathcal{F}_4$ be the projection of $\mathcal{F}_3$ onto T.

□

The Standard Example

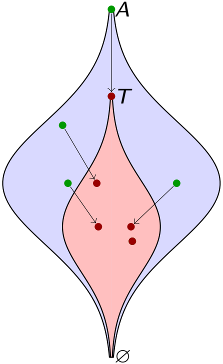
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \le 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).



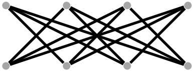
- Let $\mathcal{F}_4$ be the projection of $\mathcal{F}_3$ onto T.
- By the Projection Lemma,

$$\ell_T^-(\mathcal{F}_4) \ge \frac{|T|+1}{|A|+1} \cdot \ell_A^-(\mathcal{F}_3)$$

□

The Standard Example

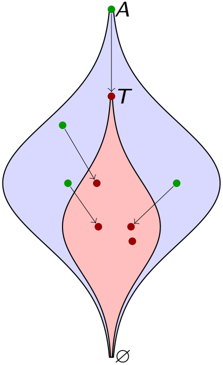
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).



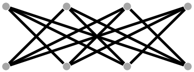
- Let $\mathcal{F}_4$ be the projection of $\mathcal{F}_3$ onto T.
- By the Projection Lemma,

$$\ell_T^-(\mathcal{F}_4) \geq \frac{|T|+1}{|A|+1} \cdot \ell_A^-(\mathcal{F}_3)$$
$$> \gamma \cdot \frac{2r-1}{\gamma}$$

□

The Standard Example

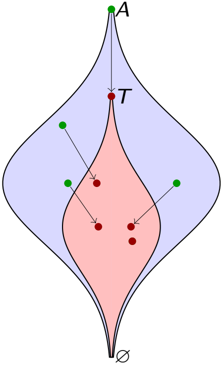
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).



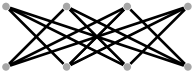
- Let $\mathcal{F}_4$ be the projection of $\mathcal{F}_3$ onto T.
- By the Projection Lemma,

$$\ell_T^-(\mathcal{F}_4) \geq \frac{|T|+1}{|A|+1} \cdot \ell_A^-(\mathcal{F}_3)$$
$$> \gamma \cdot \frac{2r-1}{\gamma}$$
$$> 2r - 1$$

□

The Standard Example

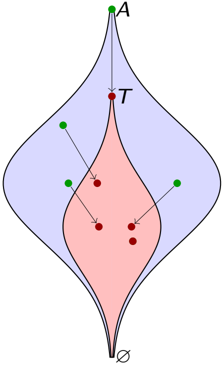
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

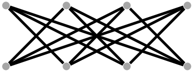


- By the Private Elements Lemma, find sets $T_1, \ldots, T_r \in \mathcal{F}_4$ and elements $t_1, \ldots, t_r \in T$ with $t_i \in T_j$ if and only if $i \neq j$.

□

The Standard Example

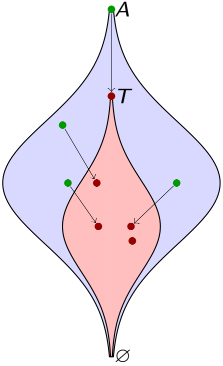
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

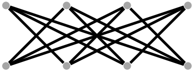


- By the Private Elements Lemma, find sets $T_1, \ldots, T_r \in \mathcal{F}_4$ and elements $t_1, \ldots, t_r \in T$ with $t_i \in T_j$ if and only if $i \neq j$.
- Each T_i extends to a set $A_i \in \mathcal{F}_3$ with $A_i \subseteq A$.

□

The Standard Example

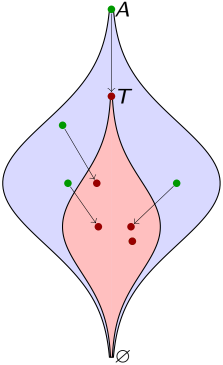
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

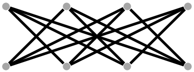


- By the Private Elements Lemma, find sets $T_1, \ldots, T_r \in \mathcal{F}_4$ and elements $t_1, \ldots, t_r \in T$ with $t_i \in T_j$ if and only if $i \neq j$.
- Each T_i extends to a set $A_i \in \mathcal{F}_3$ with $A_i \subseteq A$.
- Still, $t_i \in A_j$ if and only if $i \neq j$.

□

The Standard Example

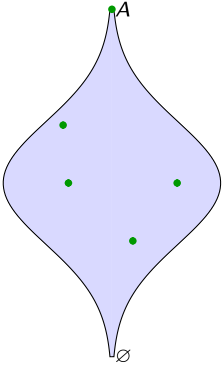
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2 \ln 2 \cdot r} + 2 \ln 2$$

Proof (sketch).

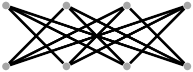


- By the Private Elements Lemma, find sets $T_1, \ldots, T_r \in \mathcal{F}_4$ and elements $t_1, \ldots, t_r \in T$ with $t_i \in T_j$ if and only if $i \neq j$.
- Each T_i extends to a set $A_i \in \mathcal{F}_3$ with $A_i \subseteq A$.
- Still, $t_i \in A_j$ if and only if $i \neq j$.

□

The Standard Example

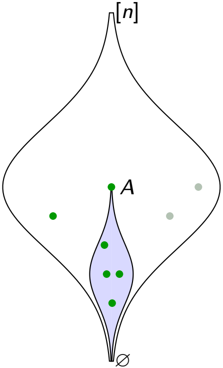
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

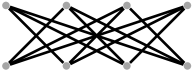


- By the Private Elements Lemma, find sets $T_1, \ldots, T_r \in \mathcal{F}_4$ and elements $t_1, \ldots, t_r \in T$ with $t_i \in T_j$ if and only if $i \neq j$.
- Each T_i extends to a set $A_i \in \mathcal{F}_3$ with $A_i \subseteq A$.
- Still, $t_i \in A_j$ if and only if $i \neq j$.

□

The Standard Example

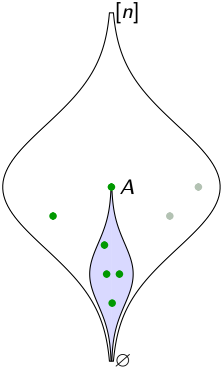
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

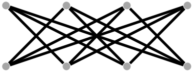


- Choose $B_j \in \mathcal{F}_1$ so that $A_j - B_j = \{t_j\}$.

□

The Standard Example

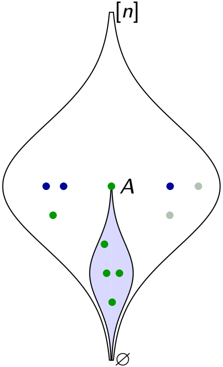
Theorem



$\mathcal{S}_4$

$$\lambda^*(\mathcal{S}_r) \leq 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

Proof (sketch).

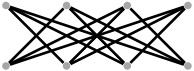


- Choose $B_j \in \mathcal{F}_1$ so that $A_j - B_j = \{t_j\}$.

□

The Standard Example

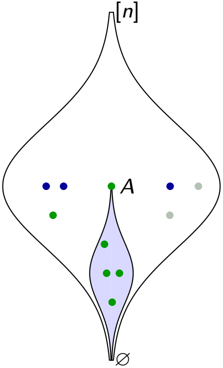
Theorem



$$\lambda^*(\mathcal{S}_r) \le 4r + 4\sqrt{2\ln 2 \cdot r} + 2\ln 2$$

$\mathcal{S}_4$

Proof (sketch).



- Choose $B_j \in \mathcal{F}_1$ so that $A_j - B_j = \{t_j\}$.
- $\{A_1, \ldots, A_r\}$ and $\{B_1, \ldots, B_r\}$ form $\mathcal{S}_r$.

□

Open Problems

Conjecture

Always $\lambda^*(P)$ is finite.

Open Problems

Conjecture

Always $\lambda^*(P)$ is finite.

- Every poset is an induced subposet of a sufficiently large boolean lattice.

Open Problems

Conjecture

Always $\lambda^*(P)$ is finite.

- Every poset is an induced subposet of a sufficiently large boolean lattice.
- $\lambda^*(2^{[3]}) \leq \lambda^*(\mathcal{S}_3) + 2 \leq 23.55$.

Open Problems

Conjecture

Always $\lambda^*(P)$ is finite.

- Every poset is an induced subposet of a sufficiently large boolean lattice.
- $\lambda^*(2^{[3]}) \leq \lambda^*(\mathcal{S}_3) + 2 \leq 23.55$.

Problem

Find a bound on $\lambda^*(2^{[4]})$.

Open Problems

Conjecture

Always $\lambda^*(P)$ is finite.

- Every poset is an induced subposet of a sufficiently large boolean lattice.
- $\lambda^*(2^{[3]}) \leq \lambda^*(\mathcal{S}_3) + 2 \leq 23.55$.

Problem

Find a bound on $\lambda^*(2^{[4]})$.

Thank You.