Chopping Celery and the Lattice of Integer Partitions

Thao Do and Bill Sands

University of Calgary

June 20, 2012
Suppose we are given a finite set of celery sticks of positive integer lengths.

Answer: (J. Ginsburg and S, 2000) At each step, choose the $w$ longest nontrivial (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than $w$ of them, and chop these all in half or as nearly in half as possible.
Suppose we are given a finite set of celery sticks of positive integer lengths.

We wish to chop these sticks into unit-length pieces, using a knife that can cut up to $w$ sticks at a time, where $w$ is a fixed positive integer (called the *width* of the knife).

Answer: (J. Ginsburg and S, 2000) At each step, choose the $w$ longest nontrivial (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than $w$ of them, and chop these all in half or as nearly in half as possible.
Suppose we are given a finite set of celery sticks of positive integer lengths.

We wish to chop these sticks into unit-length pieces, using a knife that can cut up to \( w \) sticks at a time, where \( w \) is a fixed positive integer (called the *width* of the knife).

How should we proceed in order to chop up the sticks using as few cuts as possible?

Answer: (J. Ginsburg and S., 2000) At each step, choose the \( w \) longest nontrivial (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than \( w \) of them, and chop these all in half or as nearly in half as possible.
Suppose we are given a finite set of celery sticks of positive integer lengths.

We wish to chop these sticks into unit-length pieces, using a knife that can cut up to \( w \) sticks at a time, where \( w \) is a fixed positive integer (called the width of the knife).

How should we proceed in order to chop up the sticks using as few cuts as possible?

Answer: (J. Ginsburg and S, 2000) At each step, choose the \( w \) longest nontrivial (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than \( w \) of them, and chop these all in half or as nearly in half as possible.
We will identify a set of $k$ sticks with an infinite non-increasing sequence $S$ of positive integers, where the first $k$ integers in $S$ represent the lengths of the sticks, and the remaining members of $S$ are all 1’s.
We will identify a set of $k$ sticks with an infinite non-increasing sequence $S$ of positive integers, where the first $k$ integers in $S$ represent the lengths of the sticks, and the remaining members of $S$ are all 1’s.

The set of all such sequences $S$ will be denoted $\mathcal{S}$. 
We will identify a set of \( k \) sticks with an infinite non-increasing sequence \( S \) of positive integers, where the first \( k \) integers in \( S \) represent the lengths of the sticks, and the remaining members of \( S \) are all 1’s.

The set of all such sequences \( S \) will be denoted \( \mathcal{S} \).

Note that the addition (or deletion) of 1’s (which represent trivial sticks not needing to be cut) at the end of any \( S \in \mathcal{S} \) will not affect the number of chops needed.
We will identify a set of \( k \) sticks with an infinite non-increasing sequence \( S \) of positive integers, where the first \( k \) integers in \( S \) represent the lengths of the sticks, and the remaining members of \( S \) are all 1’s.

The set of all such sequences \( S \) will be denoted \( \mathcal{S} \).

Note that the addition (or deletion) of 1’s (which represent trivial sticks not needing to be cut) at the end of any \( S \in \mathcal{S} \) will not affect the number of chops needed.

Thus, for example, \((5, 2, 2, 1, 1, \ldots)\) will usually be denoted \((5, 2, 2)\).
For each $S \in \mathcal{S}$, define the *chop vector* of $S$ by

$$v_S = (v_1, v_2, v_3, \ldots)$$

where, for each integer $w \geq 1$, $v_w$ is the minimum number of cuts needed to chop $S$ into unit pieces given a knife which can cut up to $w$ pieces at a time.
For each \( S \in \mathcal{S} \), define the \textit{chop vector} of \( S \) by

\[
v_S = (v_1, v_2, v_3, \ldots)
\]

where, for each integer \( w \geq 1 \), \( v_w \) is the minimum number of cuts needed to chop \( S \) into unit pieces given a knife which can cut up to \( w \) pieces at a time.

Note that \( v_1 \) is the number of cuts required to chop all nontrivial sticks in \( S \) into units, one stick at a time, and so \( v_1 = \sum_{s \in S} (s - 1) \).
For each $S \in \mathcal{S}$, define the \textit{chop vector} of $S$ by

$$v_S = (v_1, v_2, v_3, \ldots)$$

where, for each integer $w \geq 1$, $v_w$ is the minimum number of cuts needed to chop $S$ into unit pieces given a knife which can cut up to $w$ pieces at a time.

Note that $v_1$ is the number of cuts required to chop all nontrivial sticks in $S$ into units, one stick at a time, and so $v_1 = \sum_{s \in S} (s - 1)$.

Also, the $v_i$'s are non-increasing and non-negative integers, and so $v_S$ is eventually constant.
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:

\[
(7, 3, 2) \rightarrow (4, 3, 2, 2, 1)
\]
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:

$$(7, 3, 2) \rightarrow (4, 3, 2, 2, 1)$$

$$\rightarrow (2, 2, 2, 2, 1, 1)$$
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:

$$(7, 3, 2) \rightarrow (4, 3, 2, 2, 1)$$

$$(2, 2, 2, 2, 2, 1, 1)$$

$$(2, 2, 2, 1, 1, 1, 1, 1)$$
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:

$$(7, 3, 2) \rightarrow (4, 3, 2, 2, 1)$$

$$\rightarrow (2, 2, 2, 2, 1, 1)$$

$$\rightarrow (2, 2, 2, 1, 1, 1, 1, 1, 1)$$

$$\rightarrow (2, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$
For example, consider $S = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $S$ up in five steps as follows:

$$(7, 3, 2) \rightarrow (4, 3, 2, 2, 1)$$
$$\rightarrow (2, 2, 2, 2, 2, 1, 1)$$
$$\rightarrow (2, 2, 2, 1, 1, 1, 1, 1, 1)$$
$$\rightarrow (2, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$
$$\rightarrow (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).$$
Ignoring trivial sticks, we would write this dissection of $S = (7, 3, 2)$ as

$$(7, 3, 2) \rightarrow (4, 3, 2, 2) \rightarrow (2, 2, 2, 2, 2) \rightarrow (2, 2, 2) \rightarrow (2) \rightarrow \emptyset.$$
Ignoring trivial sticks, we would write this dissection of $S = (7, 3, 2)$ as

$$(7, 3, 2) \rightarrow (4, 3, 2, 2) \rightarrow (2, 2, 2, 2, 2) \rightarrow (2, 2, 2) \rightarrow (2) \rightarrow \emptyset.$$  

But with a knife of width $w = 3$, the chopping up takes only three steps:

$$(7, 3, 2) \rightarrow (4, 3, 2) \rightarrow (2, 2, 2) \rightarrow \emptyset.$$
Ignoring trivial sticks, we would write this dissection of $S = (7, 3, 2)$ as

$$(7, 3, 2) \rightarrow (4, 3, 2, 2) \rightarrow (2, 2, 2, 2, 2) \rightarrow (2, 2, 2) \rightarrow (2) \rightarrow \emptyset.$$ 

But with a knife of width $w = 3$, the chopping up takes only three steps:

$$(7, 3, 2) \rightarrow (4, 3, 2) \rightarrow (2, 2, 2) \rightarrow \emptyset.$$ 

Moreover, it is easy to see that, for any width $w \geq 3$, at least three cuts will be necessary to reduce the stick of length 7 down to unit pieces.
Ignoring trivial sticks, we would write this dissection of $S = (7, 3, 2)$ as

$$(7, 3, 2) \rightarrow (4, 3, 2, 2) \rightarrow (2, 2, 2, 2) \rightarrow (2, 2, 2) \rightarrow (2) \rightarrow \emptyset.$$  

But with a knife of width $w = 3$, the chopping up takes only three steps:

$$(7, 3, 2) \rightarrow (4, 3, 2) \rightarrow (2, 2, 2) \rightarrow \emptyset.$$  

Moreover, it is easy to see that, for any width $w \geq 3$, at least three cuts will be necessary to reduce the stick of length 7 down to unit pieces. Thus

$$v_{(7,3,2)} = (9, 5, 3, 3, \ldots).$$
Ignoring trivial sticks, we would write this dissection of $S = (7, 3, 2)$ as

$$(7, 3, 2) \rightarrow (4, 3, 2, 2) \rightarrow (2, 2, 2, 2, 2) \rightarrow (2, 2, 2) \rightarrow (2) \rightarrow \emptyset.$$ 

But with a knife of width $w = 3$, the chopping up takes only three steps:

$$(7, 3, 2) \rightarrow (4, 3, 2) \rightarrow (2, 2, 2) \rightarrow \emptyset.$$ 

Moreover, it is easy to see that, for any width $w \geq 3$, at least three cuts will be necessary to reduce the stick of length 7 down to unit pieces. Thus

$$\nu_{(7,3,2)} = (9, 5, 3, 3, \ldots).$$

But what does all this have to do with partially ordered sets??
If we deduct 1 from each entry in a sequence $S \in \mathcal{S}$, we obtain an infinite non-increasing sequence $S'$ of non-negative integers, only finitely many of which are nonzero.
If we deduct 1 from each entry in a sequence \( S \in \mathcal{S} \), we obtain an infinite non-increasing sequence \( S' \) of non-negative integers, only finitely many of which are nonzero.

Thus we will consider \( S' \) as a partition of the positive integer

\[
\sum_{s' \in S'} s' = \sum_{s \in S} (s - 1).
\]
If we deduct 1 from each entry in a sequence $S \in \mathcal{S}$, we obtain an infinite non-increasing sequence $S'$ of non-negative integers, only finitely many of which are nonzero.

Thus we will consider $S'$ as a partition of the positive integer $
\sum_{s' \in S'} s' = \sum_{s \in S} (s - 1).
$

Therefore $\{S' : S \in \mathcal{S}\}$ forms the set $\mathcal{P}$ of all integer partitions.
If we deduct 1 from each entry in a sequence \( S \in \mathcal{S} \), we obtain an infinite non-increasing sequence \( S' \) of non-negative integers, only finitely many of which are nonzero.

Thus we will consider \( S' \) as a partition of the positive integer

\[
\sum_{s' \in S'} s' = \sum_{s \in S} (s - 1).
\]

Therefore \( \{S' : S \in \mathcal{S}\} \) forms the set \( \mathcal{P} \) of all integer partitions.

Furthermore \( \mathcal{P} \) can be given a natural partial ordering \( \leq \) called dominance ordering (or majorization) as follows. For integer partitions \( S = (s_1, s_2, \ldots) \) and \( T = (t_1, t_2, \ldots) \) in \( \mathcal{P} \), put \( S \leq T \) if and only if

\[
\sum_{i=1}^{j} s_i \leq \sum_{i=1}^{j} t_i \text{ for all } j \geq 1.
\]
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the \textit{lattice of integer partitions}. 
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the *lattice of integer partitions*.

- Brylawski (1973)
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the *lattice of integer partitions*.

- Brylawski (1973)
- Baransky and Koroleva (2008)
- Latapy and Phan (2009)
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the *lattice of integer partitions*.

- Brylawski (1973)
- Baransky and Koroleva (2008)
- Latapy and Phan (2009)

We can similarly define dominance ordering $\leq$ on the set $\mathcal{S}$ of all sets of sticks.
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the *lattice of integer partitions*.

- Brylawski (1973)
- Baransky and Koroleva (2008)
- Latapy and Phan (2009)

We can similarly define dominance ordering $\leq$ on the set $\mathcal{I}$ of all sets of sticks. Then $(\mathcal{I}, \leq)$ becomes a lattice, clearly isomorphic to $\mathcal{P}$, via the renaming $S \rightarrow S'$. 
Then $\mathcal{P} = (\mathcal{P}, \leq)$ is a lattice called the **lattice of integer partitions**.

- Brylawski (1973)
- Baransky and Koroleva (2008)
- Latapy and Phan (2009)

We can similarly define dominance ordering $\leq$ on the set $\mathcal{I}$ of all sets of sticks. Then $(\mathcal{I}, \leq)$ becomes a lattice, clearly isomorphic to $\mathcal{P}$, via the renaming $S \rightarrow S'$.

Thao Do: Master’s Thesis on integer partitions (U of C, 2009)
Dominance ordering $\leq$ on the lattice $\mathcal{S}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{S}$, $S < T$ if
Dominance ordering $\leq$ on the lattice $\mathcal{I}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{I}$, $S < T$ if

(i) $n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $i \in \{1, 2, \ldots, m\}$, or

(ii) $n = m$ and there exists $1 \leq j < k \leq m$ so that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or $k$. Considering $S$ and $T$ as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths, (i) is equivalent to $T = S \cup \{2\}$, and (ii) is equivalent to $T = (S - \{x, y\}) \cup \{x - 1, y + 1\}$ for some $x, y$ in $S$ satisfying $2 \leq x \leq y$. 
Dominance ordering $\leq$ on the lattice $\mathcal{S}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{S}$, $S < T$ if

(i) $n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $i \in \{1, 2, \ldots, m\}$, or

(ii) $n = m$ and there exists $1 \leq j < k \leq m$ so that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or $k$. 

Considering $S$ and $T$ as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths, (i) is equivalent to $T = S \cup \{2\}$, and (ii) is equivalent to $T = (S - \{x, y\}) \cup \{x - 1, y + 1\}$ for some $x, y$ in $S$ satisfying $2 \leq x \leq y$. 
Dominance ordering $\leq$ on the lattice $\mathcal{S}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{S}$, $S < T$ if

(i) $n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $i \in \{1, 2, \ldots, m\}$, or

(ii) $n = m$ and there exists $1 \leq j < k \leq m$ so that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or $k$.

Considering $S$ and $T$ as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths,
Dominance ordering $\leq$ on the lattice $\mathcal{L}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{L}$, $S \prec T$ if

(i) $n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $i \in \{1, 2, \ldots, m\}$, or

(ii) $n = m$ and there exists $1 \leq j < k \leq m$ so that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or $k$.

Considering $S$ and $T$ as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths,

- (i) is equivalent to $T = S \cup \{2\}$, and
Dominance ordering $\leq$ on the lattice $\mathcal{L}$ is the transitive and reflexive closure of the following two types of relations: for $S = (s_1, s_2, \ldots, s_m)$ and $T = (t_1, t_2, \ldots, t_n)$ in $\mathcal{L}$, $S < T$ if

(i) $n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $i \in \{1, 2, \ldots, m\}$, or

(ii) $n = m$ and there exists $1 \leq j < k \leq m$ so that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or $k$.

Considering $S$ and $T$ as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths,

- (i) is equivalent to $T = S \cup \{2\}$, and

- (ii) is equivalent to $T = (S - \{x, y\}) \cup \{x - 1, y + 1\}$ for some $x, y$ in $S$ satisfying $2 \leq x \leq y$. 
For instance, \( (4, 3) < (4, 3, 2) \) in the Figure is an example of the first kind of relation above,

\[ (4, 3), (4, 3, 2) < (5, 2, 2), (4, 3, 2) < (4, 4, 1) = (4, 4) \]

in the transitive closure.

The family of all chop vectors, considered as elements of the direct product \( \mathbb{N}^\omega \), can be naturally ordered componentwise; that is, for all \( S, T \in \mathbb{N}^\omega \), \( v_S \leq v_T \) if and only if \( (v_S)_i \leq (v_T)_i \) for all \( i \).
For instance, $(4, 3) < (4, 3, 2)$ in the Figure is an example of the first kind of relation above, while $(4, 3, 2) < (5, 2, 2)$ and $(4, 3, 2) < (4, 4, 1) = (4, 4)$ are examples of the second kind.
For instance, \((4, 3) < (4, 3, 2)\) in the Figure is an example of the first kind of relation above, while \((4, 3, 2) < (5, 2, 2)\) and \((4, 3, 2) < (4, 4, 1) = (4, 4)\) are examples of the second kind. Thus \((4, 3) < (5, 2, 2)\) and \((4, 3) < (4, 4)\) in the transitive closure.
For instance, \((4, 3) < (4, 3, 2)\) in the Figure is an example of the first kind of relation above, while \((4, 3, 2) < (5, 2, 2)\) and \((4, 3, 2) < (4, 4, 1) = (4, 4)\) are examples of the second kind. Thus \((4, 3) < (5, 2, 2)\) and \((4, 3) < (4, 4)\) in the transitive closure.

The family of all chop vectors, considered as elements of the direct product \(\mathbb{N}\omega\), can be naturally ordered componentwise;
For instance, \((4, 3) < (4, 3, 2)\) in the Figure is an example of the first kind of relation above, while \((4, 3, 2) < (5, 2, 2)\) and \((4, 3, 2) < (4, 4, 1) = (4, 4)\) are examples of the second kind. Thus \((4, 3) < (5, 2, 2)\) and \((4, 3) < (4, 4)\) in the transitive closure.

The family of all chop vectors, considered as elements of the direct product \(\mathbb{N}^\omega\), can be naturally ordered componentwise; that is, for all \(S, T \in \mathcal{S}\), \(\mathbf{v}_S \leq \mathbf{v}_T\) if and only if \((\mathbf{v}_S)_i \leq (\mathbf{v}_T)_i\) for all \(i\).
Let $\phi : \mathcal{I} \to \mathbb{N}^\omega$ defined by $\phi(S) = v_S$ for all $S \in \mathcal{I}$. 
Let $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ defined by $\phi(S) = v_S$ for all $S \in \mathcal{S}$.

**Theorem**

(T. Do, B. Sands) $\phi$ is order preserving; that is, for all $S, \mathcal{T} \in \mathcal{S}$ with $S \leq \mathcal{T}$, $v_S \leq v_{\mathcal{T}}$. 
Let $\phi : \mathcal{S} \to \mathbb{N}^\omega$ defined by $\phi(S) = v_S$ for all $S \in \mathcal{S}$.

**Theorem**

(T. Do, B. Sands) $\phi$ is order preserving; that is, for all $S, T \in \mathcal{S}$ with $S \leq T$, $v_S \leq v_T$.

The proof is a slightly tricky induction on the number of steps required to completely chop up a set of sticks.
Note: $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ given by $\phi(S) = v_S$ is order-preserving, but is not a lattice homomorphism.
Note: $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ given by $\phi(S) = v_S$ is order-preserving, but is not a lattice homomorphism.

For example, let $S = (3)$ and $T = (2, 2, 2)$. Then
Note: $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ given by $\phi(S) = v_S$ is order-preserving, but is not a lattice homomorphism.

For example, let $S = (3)$ and $T = (2, 2, 2)$. Then

$$\phi(3) = v(3) = (2, 2, \ldots) \quad \text{and} \quad \phi(2, 2, 2) = v_{(2,2,2)} = (3, 2, 1, 1, \ldots).$$
Note: \( \phi : \mathcal{L} \rightarrow \mathbb{N}^\omega \) given by \( \phi(S) = v_S \) is order-preserving, but is not a lattice homomorphism.

For example, let \( S = (3) \) and \( T = (2, 2, 2) \). Then

\[
\phi(3) = v_{(3)} = (2, 2, \ldots) \quad \text{and} \quad \phi(2, 2, 2) = v_{(2,2,2)} = (3, 2, 1, 1, \ldots).
\]

Thus \( \phi(S) \wedge \phi(T) = (2, 2, 1, 1, \ldots) \).
Note: \( \phi : S \rightarrow \mathbb{N}^\omega \) given by \( \phi(S) = v_S \) is order-preserving, but is not a lattice homomorphism.

For example, let \( S = (3) \) and \( T = (2, 2, 2) \). Then

\[
\phi(3) = v_{(3)} = (2, 2, \ldots) \quad \text{and} \quad \phi(2, 2, 2) = v_{(2,2,2)} = (3, 2, 1, 1, \ldots).
\]

Thus \( \phi(S) \land \phi(T) = (2, 2, 1, 1, \ldots) \).

However, \( S \land T = (2, 2) \), and
Note: $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ given by $\phi(S) = v_S$ is order-preserving, but is not a lattice homomorphism.

For example, let $S = (3)$ and $T = (2, 2, 2)$. Then

$$\phi(3) = v_3 = (2, 2, \ldots) \quad \text{and} \quad \phi(2, 2, 2) = v_{(2,2,2)} = (3, 2, 1, 1, \ldots).$$

Thus $\phi(S) \wedge \phi(T) = (2, 2, 1, 1, \ldots)$.

However, $S \wedge T = (2, 2)$, and

$$\phi(2, 2) = v_{(2,2)} = (2, 1, 1, \ldots) < \phi(S) \wedge \phi(T);$$
Note: $\phi: \mathcal{L} \to \mathbb{N}^\omega$ given by $\phi(S) = v_S$ is order-preserving, but is not a lattice homomorphism.

For example, let $S = (3)$ and $T = (2, 2, 2)$. Then

$$\phi(3) = v_{(3)} = (2, 2, \ldots)$$

and

$$\phi(2, 2, 2) = v_{(2,2,2)} = (3, 2, 1, 1, \ldots).$$

Thus $\phi(S) \wedge \phi(T) = (2, 2, 1, 1, \ldots)$.

However, $S \wedge T = (2, 2)$, and

$$\phi(2, 2) = v_{(2,2)} = (2, 1, 1, \ldots) < \phi(S) \wedge \phi(T);$$

note that when $w = 2$, the binary algorithm produces

$$(2, 2) \rightarrow \emptyset,$$

so $v_2 = 1$ in $v_{(2,2)}$. 
Note: $\phi : \mathcal{S} \rightarrow \mathbb{N}^\omega$ given by $\phi(S) = \mathbf{v}_S$ is order-preserving, but is not a lattice homomorphism.

For example, let $S = (3)$ and $T = (2, 2, 2)$. Then

$$\phi(3) = \mathbf{v}_3 = (2, 2, \ldots) \quad \text{and} \quad \phi(2, 2, 2) = \mathbf{v}_{(2,2,2)} = (3, 2, 1, 1, \ldots).$$

Thus $\phi(S) \land \phi(T) = (2, 2, 1, 1, \ldots)$.

However, $S \land T = (2, 2)$, and

$$\phi(2, 2) = \mathbf{v}_{(2,2)} = (2, 1, 1, \ldots) < \phi(S) \land \phi(T);$$

note that when $w = 2$, the binary algorithm produces

$$(2, 2) \rightarrow \emptyset,$$

so $v_2 = 1$ in $\mathbf{v}_{(2,2)}$. 
In contrast, all the joins illustrated in the Figure are in fact preserved, and we have not yet found a join that is not.
In contrast, all the joins illustrated in the Figure are in fact preserved, and we have not yet found a join that is not.

Problem

1. Is $\phi$ join preserving?
In contrast, all the joins illustrated in the Figure are in fact preserved, and we have not yet found a join that is not.

**Problem**

1. *Is \( \phi \) join preserving?*

An affirmative answer to this problem would supply an alternate proof to our Theorem.
For a vector $v$, let $\mathcal{S}(v)$ be the family of all sets $S$ of sticks whose chop vector $v_S$ equals $v$. Then $\mathcal{S}(v)$ is a convex subset of $\mathcal{S}$. That is, if $S$ and $T$ are in $\mathcal{S}(v)$ and satisfy $S < T$ in $\mathcal{S}$, and if $U$ is in $\mathcal{S}$ and satisfies $S < U < T$, then $U$ must be in $\mathcal{S}(v)$. 
For a vector \( \mathbf{v} \), let \( \mathcal{I}(\mathbf{v}) \) be the family of all sets \( S \) of sticks whose chop vector \( \mathbf{v}_S \) equals \( \mathbf{v} \).

Then \( \mathcal{I}(\mathbf{v}) \) is a convex subset of \( \mathcal{I} \).
For a vector $\mathbf{v}$, let $\mathcal{S}(\mathbf{v})$ be the family of all sets $S$ of sticks whose chop vector $\mathbf{v}_S$ equals $\mathbf{v}$.

Then $\mathcal{S}(\mathbf{v})$ is a convex subset of $\mathcal{S}$.

That is, if $S$ and $T$ are in $\mathcal{S}(\mathbf{v})$ and satisfy $S < T$ in $\mathcal{S}$, and if $U$ is in $\mathcal{S}$ and satisfies $S < U < T$, then $U$ must be in $\mathcal{S}(\mathbf{v})$. 
However, $\mathcal{I}(v)$ is not always a sublattice of $\mathcal{I}$, in particular $\mathcal{I}(v)$ is not always closed under meets.
However, $\mathcal{L}(v)$ is not always a sublattice of $\mathcal{L}$, in particular $\mathcal{L}(v)$ is not always closed under meets.

For example, let $\mathcal{S} = (7, 4)$ and $\mathcal{T} = (8, 2, 2)$.
However, $\mathcal{L}(v)$ is not always a sublattice of $\mathcal{L}$, in particular $\mathcal{L}(v)$ is not always closed under meets.

For example, let $S = (7, 4)$ and $T = (8, 2, 2)$. Then

$$v_S = (9, 5, 4, 3, 3, \ldots) = v_T,$$

But $S \land T = (7, 4) \land (8, 2, 2) = (7, 3, 2)$, and $v_{(7, 3, 2)} = (9, 5, 4, 3, 3, \ldots)$, so $S \land T \notin S(v)$. 

17/22 Thao Do and Bill Sands (University of Calgary) Chopping Celery and the Lattice of Integer Partitions June 20, 2012 17/22
However, $\mathcal{I}(\mathbf{v})$ is not always a sublattice of $\mathcal{I}$, in particular $\mathcal{I}(\mathbf{v})$ is not always closed under meets.

For example, let $\mathcal{S} = (7, 4)$ and $\mathcal{T} = (8, 2, 2)$. Then

$$\mathbf{v}_\mathcal{S} = (9, 5, 4, 3, 3, \ldots) = \mathbf{v}_\mathcal{T},$$

so

$$\mathcal{S}, \mathcal{T} \in \mathcal{I}(\mathbf{v}) \quad \text{where} \quad \mathbf{v} = (9, 5, 4, 3, 3, \ldots).$$
However, $\mathcal{I}(v)$ is not always a sublattice of $\mathcal{I}$, in particular $\mathcal{I}(v)$ is not always closed under meets.

For example, let $S = (7, 4)$ and $T = (8, 2, 2)$. Then

$$v_S = (9, 5, 4, 3, 3, \ldots) = v_T,$$

so

$$S, T \in \mathcal{I}(v) \quad \text{where} \quad v = (9, 5, 4, 3, 3, \ldots).$$

But $S \land T = (7, 4) \land (8, 2, 2) = (7, 3, 2)$,
However, $\mathcal{I}(v)$ is not always a sublattice of $\mathcal{I}$, in particular $\mathcal{I}(v)$ is not always closed under meets.

For example, let $S = (7, 4)$ and $T = (8, 2, 2)$. Then

$$v_S = (9, 5, 4, 3, 3, \ldots) = v_T,$$

so

$$S, T \in \mathcal{I}(v) \text{ where } v = (9, 5, 4, 3, 3, \ldots).$$

But $S \land T = (7, 4) \land (8, 2, 2) = (7, 3, 2)$, and

$$v_{(7,3,2)} = (9, 5, 3, 3, \ldots).$$
However, $\mathcal{I}(\mathbf{v})$ is not always a sublattice of $\mathcal{I}$, in particular $\mathcal{I}(\mathbf{v})$ is not always closed under meets.

For example, let $S = (7,4)$ and $T = (8,2,2)$. Then

$$\mathbf{v}_S = (9,5,4,3,3,\ldots) = \mathbf{v}_T,$$

so

$$S, T \in \mathcal{I}(\mathbf{v}) \quad \text{where} \quad \mathbf{v} = (9,5,4,3,3,\ldots).$$

But $S \wedge T = (7,4) \wedge (8,2,2) = (7,3,2)$, and

$$\mathbf{v}_{(7,3,2)} = (9,5,3,3,\ldots),$$

so $S \wedge T \notin \mathcal{I}(\mathbf{v})$. 
Problem

2 For all vectors \( v \), is \( \mathcal{I}(v) \) closed under joins?
Problem

2 For all vectors $\mathbf{v}$, is $\mathcal{I}(\mathbf{v})$ closed under joins?

An affirmative answer to Problem 1 would give an affirmative answer to Problem 2 as well.
Here is another question suggested by the convex subsets $\mathcal{I}(\mathbf{v})$. Call a set $S$ of sticks *lonely* if $\mathcal{I}(\mathbf{v}_S) = \{S\}$, that is, if $S$ is the only element of $\mathcal{I}$ having that particular chop vector.
Here is another question suggested by the convex subsets $\mathcal{I}(v)$. Call a set $S$ of sticks *lonely* if $\mathcal{I}(v_S) = \{S\}$, that is, if $S$ is the only element of $\mathcal{I}$ having that particular chop vector.

It is easy to see that for $S = (2, 2, \ldots, 2)$ (which we abbreviate as $(2^n)$ if there are $n$ 2’s), its chop vector $v_S$ satisfies $v_1 = n$ and $v_m = 1$ for all $m \geq n$. 
Here is another question suggested by the convex subsets $\mathcal{I}(v)$. Call a set $S$ of sticks *lonely* if $\mathcal{I}(v_S) = \{S\}$, that is, if $S$ is the only element of $\mathcal{I}$ having that particular chop vector.

It is easy to see that for $S = (2, 2, \ldots, 2)$ (which we abbreviate as $(2^n)$ if there are $n$ 2’s), its chop vector $v_S$ satisfies $v_1 = n$ and $v_m = 1$ for all $m \geq n$.

Moreover the sequences $S = (2^n)$ for integers $n \geq 1$ are the only elements $S \in \mathcal{I}$ so that $v_S$ is eventually 1.
Here is another question suggested by the convex subsets $\mathcal{I}(v)$. Call a set $S$ of sticks *lonely* if $\mathcal{I}(v_S) = \{S\}$, that is, if $S$ is the only element of $\mathcal{I}$ having that particular chop vector.

It is easy to see that for $S = (2, 2, \ldots, 2)$ (which we abbreviate as $(2^n)$ if there are $n$ 2’s), its chop vector $v_S$ satisfies $v_1 = n$ and $v_m = 1$ for all $m \geq n$.

Moreover the sequences $S = (2^n)$ for integers $n \geq 1$ are the only elements $S \in \mathcal{I}$ so that $v_S$ is eventually 1.

Thus $(2^n)$ is lonely for all integers $n \geq 1$. 
Also, of the elements \( S \in \mathcal{S} \) of the form \( S = (n) \) for integers \( n \leq 12 \), the following are lonely:

\( n = 2, 3, 5, 7, 8, 9, 11 \).

And there is more: among the remaining elements shown in the Figure, \((4, 4), (4, 4, 3)\) and \((8, 2)\) are lonely.

(Incidentally, this last example shows that a lonely element need not be a join-irreducible element of the lattice \( \mathcal{S} \).)
Also, of the elements $S \in \mathcal{S}$ of the form $S = (n)$ for integers $n \leq 12$, the following are lonely: $n = 2, 3, 5, 7, 8, 9, 11$. 
Also, of the elements \( S \in \mathcal{I} \) of the form \( S = (n) \) for integers \( n \leq 12 \), the following are lonely: \( n = 2, 3, 5, 7, 8, 9, 11 \).

And there is more: among the remaining elements shown in the Figure, \((4, 4), (4, 4, 3)\) and \((8, 2)\) are lonely.
Also, of the elements $S \in \mathcal{I}$ of the form $S = (n)$ for integers $n \leq 12$, the following are lonely: $n = 2, 3, 5, 7, 8, 9, 11$.

And there is more: among the remaining elements shown in the Figure, $(4, 4), (4, 4, 3)$ and $(8, 2)$ are lonely.

(Incidentally, this last example shows that a lonely element need not be a join-irreducible element of the lattice $\mathcal{I}$.)
Also, of the elements \( S \in \mathcal{I} \) of the form \( S = (n) \) for integers \( n \leq 12 \), the following are lonely: \( n = 2, 3, 5, 7, 8, 9, 11 \).

And there is more: among the remaining elements shown in the Figure, \((4, 4), (4, 4, 3)\) and \((8, 2)\) are lonely.

(Incidentally, this last example shows that a lonely element need not be a join-irreducible element of the lattice \( \mathcal{I} \).)

**Problem**

3 **Characterize all lonely elements of \( \mathcal{I} \).**


