Chopping Celery and the Lattice of Integer Partitions

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How should we proceed in order to chop up the sticks using as few cuts as possible?

Answer: (J. Ginsburg and S, 2000) At each step, choose the w longest nontrivial (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than w of them, and chop these all in half or as nearly in half as possible.

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Note that the addition (or deletion) of 1's (which represent trivial sticks not needing to be cut) at the end of any $\mathcal{S} \in \mathscr{S}$ will not affect the number of chops needed.

Thus, for example, $(5, 2, 2, 1, 1, \ldots)$ will usually be denoted $(5, 2, 2)$.

For each $\mathcal{S} \in \mathscr{S}$, define the *chop vector* of $\mathcal{S}$ by

$$\mathbf{v}_{\mathcal{S}} = (v_1, v_2, v_3, \dots)$$

where, for each integer $w \geq 1$, v_w is the minimum number of cuts needed to chop $\mathcal{S}$ into unit pieces given a knife which can cut up to w pieces at a time.

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Also, the v_i's are non-increasing and non-negative integers, and so $\mathbf{v}_{\mathcal{S}}$ is eventually constant.

For example, consider $\mathcal{S} = (7, 3, 2)$. Then $v_1 = 6 + 2 + 1 = 9$, and $v_2 = 5$ because, with a knife of width $w = 2$, the binary algorithm would cut $\mathcal{S}$ up in five steps as follows:

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&\rightarrow (1,1,1,1,1,1,1,1,1,1,1,1).
\end{aligned}$$

Ignoring trivial sticks, we would write this dissection of $\mathcal{S} = (7, 3, 2)$ as

$$(7, 3, 2) \to (4, 3, 2, 2) \to (2, 2, 2, 2, 2) \to (2, 2, 2) \to (2) \to \emptyset.$$

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But what does all this have to do with partially ordered sets??

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Furthermore $\mathscr{P}$ can be given a natural partial ordering $\leq$ called *dominance ordering* (or *majorization*) as follows. For integer partitions $\mathcal{S} = (s_1, s_2, \dots)$ and $\mathcal{T} = (t_1, t_2, \dots)$ in $\mathscr{P}$, put $\mathcal{S} \leq \mathcal{T}$ if and only if $\sum_{i=1}^{j} s_i \leq \sum_{i=1}^{j} t_i$ for all $j \geq 1$.

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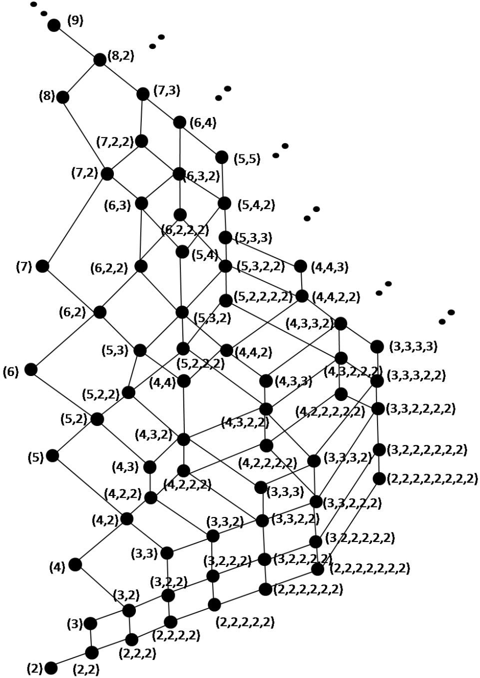
Thao Do: Master's Thesis on integer partitions (U of C, 2009)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The family of all chop vectors, considered as elements of the direct product $\mathbb{N}^\omega$, can be naturally ordered componentwise; that is, for all $\mathcal{S}, \mathcal{T} \in \mathscr{S}$, $\mathbf{v}_\mathcal{S} \leq \mathbf{v}_\mathcal{T}$ if and only if $(\mathbf{v}_\mathcal{S})_i \leq (\mathbf{v}_\mathcal{T})_i$ for all i.

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(T. Do, B. Sands) ϕ is order preserving; that is, for all $\mathcal{S}, \mathcal{T} \in \mathscr{S}$ with $\mathcal{S} \leq \mathcal{T}$, $\mathbf{v}_\mathcal{S} \leq \mathbf{v}_\mathcal{T}$.

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The proof is a slightly tricky induction on the number of steps required to completely chop up a set of sticks.

Note: $\phi : \mathscr{S} \to \mathbb{N}^\omega$ given by $\phi(\mathcal{S}) = \mathbf{v}_{\mathcal{S}}$ is order-preserving, but is **not** a lattice homomorphism.

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note that when $w = 2$, the binary algorithm produces

$$(2, 2) \to \emptyset,$$

so $v_2 = 1$ in $\mathbf{v}_{(2,2)}$.

Note: $\phi : \mathscr{S} \to \mathbb{N}^\omega$ given by $\phi(\mathcal{S}) = \mathbf{v}_\mathcal{S}$ is order-preserving, but is **not** a lattice homomorphism.

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$$\phi(3) = \mathbf{v}_{(3)} = (2,2,\ldots) \quad \text{and} \quad \phi(2,2,2) = \mathbf{v}_{(2,2,2)} = (3,2,1,1,\ldots).$$

Thus $\phi(\mathcal{S}) \wedge \phi(\mathcal{T}) = (2,2,1,1,\ldots)$.

However, $\mathcal{S} \wedge \mathcal{T} = (2,2)$, and

$$\phi(2,2) = \mathbf{v}_{(2,2)} = (2,1,1,\ldots) < \phi(\mathcal{S}) \wedge \phi(\mathcal{T});$$

note that when $w = 2$, the binary algorithm produces

$$(2,2) \to \emptyset,$$

so $v_2 = 1$ in $\mathbf{v}_{(2,2)}$.

In contrast, all the joins illustrated in the Figure are in fact preserved, and we have not yet found a join that is not.

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Problem

1 *Is ϕ join preserving?*

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An affirmative answer to this problem would supply an alternate proof to our Theorem.

For a vector $\mathbf{v}$, let $\mathscr{S}(\mathbf{v})$ be the family of all sets $\mathcal{S}$ of sticks whose chop vector $\mathbf{v}_{\mathcal{S}}$ equals $\mathbf{v}$.

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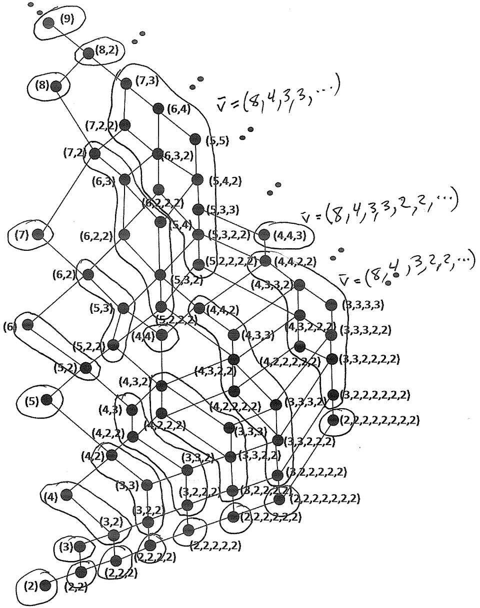
Then $\mathscr{S}(\mathbf{v})$ is a convex subset of $\mathscr{S}$.

That is, if $\mathcal{S}$ and $\mathcal{T}$ are in $\mathscr{S}(\mathbf{v})$ and satisfy $\mathcal{S} < \mathcal{T}$ in $\mathscr{S}$, and if $\mathcal{U}$ is in $\mathscr{S}$ and satisfies $\mathcal{S} < \mathcal{U} < \mathcal{T}$, then $\mathcal{U}$ must be in $\mathscr{S}(\mathbf{v})$.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so $\mathcal{S} \wedge \mathcal{T} \notin \mathscr{S}(\mathbf{v})$.

Problem

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An affirmative answer to Problem 1 would give an affirmative answer to Problem 2 as well.

Here is another question suggested by the convex subsets $\mathscr{S}(\mathbf{v})$. Call a set $\mathcal{S}$ of sticks *lonely* if $\mathscr{S}(\mathbf{v}_{\mathcal{S}}) = \{\mathcal{S}\}$, that is, if $\mathcal{S}$ is the only element of $\mathscr{S}$ having that particular chop vector.

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It is easy to see that for $\mathcal{S} = (2, 2, \ldots, 2)$ (which we abbreviate as (2^n) if there are n 2's), its chop vector $\mathbf{v}_{\mathcal{S}}$ satisfies $v_1 = n$ and $v_m = 1$ for all $m \geq n$.

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Thus (2^n) is lonely for all integers $n \geq 1$.

Also, of the elements $\mathcal{S} \in \mathscr{S}$ of the form $\mathcal{S} = (n)$ for integers $n \leq 12$, the following are lonely:

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And there is more: among the remaining elements shown in the Figure, $(4, 4), (4, 4, 3)$ and $(8, 2)$ are lonely.

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Problem

3 *Characterize all lonely elements of $\mathscr{S}$.*

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