First-Fit Coloring of Ladder-Free Posets

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Chain Partitions of Posets

- X is a set of vertices
- $\leq$ is a reflexive, transitive, antisymmetric order on X
- $P = (X, \leq)$ is a poset
- $C \subseteq X$ is a chain if its elements are pairwise comparable.
- $A \subseteq X$ is an antichain if its elements are pairwise imcomparable.
- $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ is a chain partition of P if each C_j is a chain and $X = \bigcup_{1 \leq j \leq n} C_j$.

Theorem

(Dilworth, 1950) Any poset of width w can be partitioned into w chains. Furthermore, no poset of width w can be partitioned into fewer that w chains.

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- In alternating rounds, Spoiler reveals an element of a poset to Algorithm along with all comparabilities. Algorithm builds a chain partition by assigning each element to a chain when Spoiler reveals it.
- $\mathrm{val}(w)$ is the largest integer m so that Spoiler has a poset of width at most w and order of revealing the elements that forces Algorithm to use at least m chains. Dually, it is the smallest integer n so that Algorithm may play the game indefinitely using only n chains for any poset of width w and for any order in which the elements are revealed.

Known Bounds

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- $\text{val}(w) \leq w^{3+6.5 \lg w}$ MES, Bosek, Kierstead, Krawczyk (2012)

First-Fit Chain Partitioning

- Spoiler plays as before.
- Algorithm must use a greedy strategy; i.e.: Algorithm indexes the chains he is building as $C_1, C_2, \ldots, C_n$. When Spoiler introduces a new element x, then Algorithm must assign x to C_j where j is the smallest index so that $C_j + x$ is a chain. If no such chain exists, Algorithm adds chain C_{n+1}.

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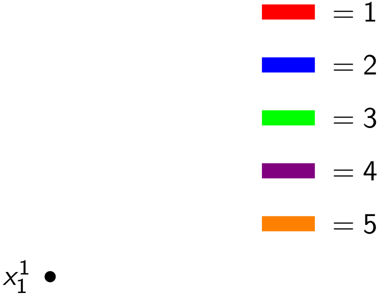

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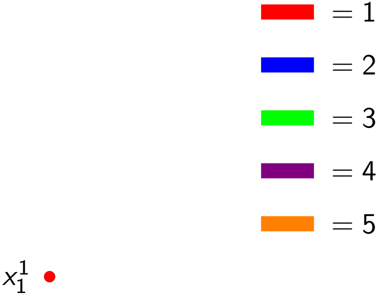

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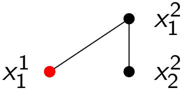

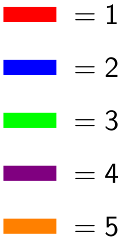

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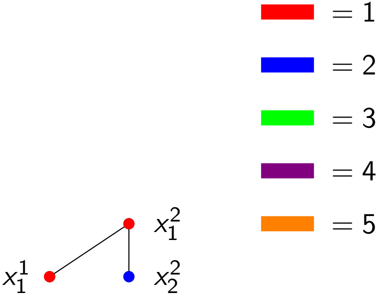

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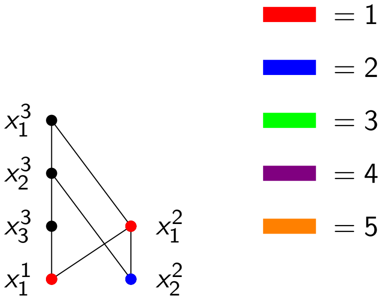

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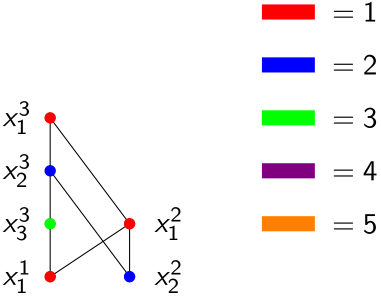

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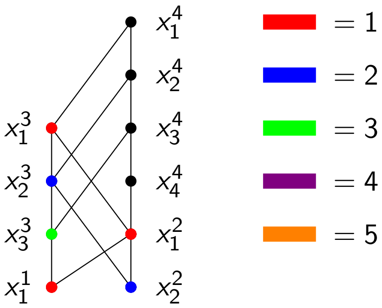

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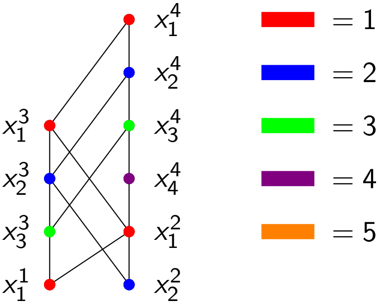

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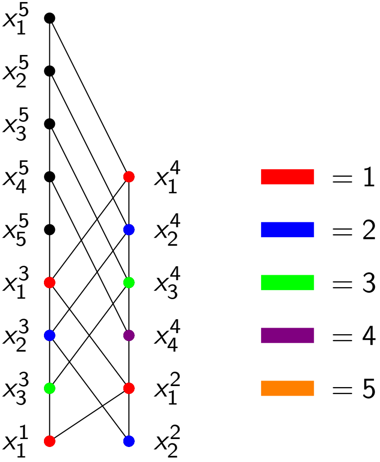

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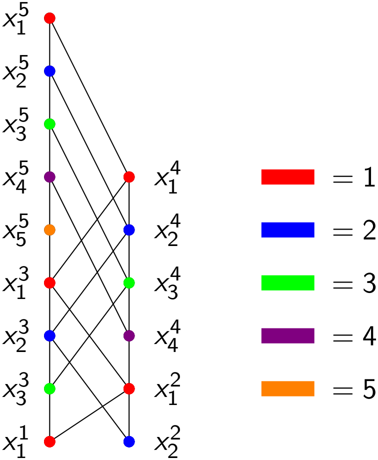

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  Let Q be a width 2 poset. If P is Q-free, then First-Fit uses a bounded number of chains to partition P on-line.
- The number of chains Spoiler can force First-Fit to use in coloring a width Q-free width w poset is is $\text{val}_{FF}(Q, w)$.

Grundy Colorings

- The function $g : P \to [n]$ is an n-Grundy coloring if:
  1. g is surjective
  2. $\{u \in P : g(u) = i\}$ is a chain
  3. If $g(v) = j$, then for each $1 \leq i < j$, there is some u with $g(u) = i$ and $u \| v$. The vertex u is a witness for v.

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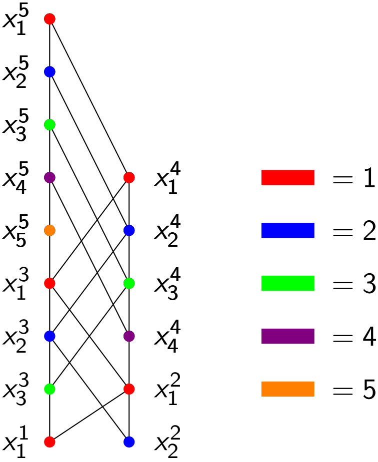
- There is a presentation of P that forces First-Fit to use n chains iff P has a n-Grundy coloring.
  - Given n-Grundy coloring g, present vertex u before v if $g(u) < g(v)$ (their order chosen arbitrarily if $g(u) = g(v)$).
  - Given a presentation order that forces $C_1, C_2, \ldots, C_n$ chains to be used, define g by $g(u) = i$ iff $u \in C_i$.

For Example ...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x_3^5
x_4^5
x_5^5
x_1^3
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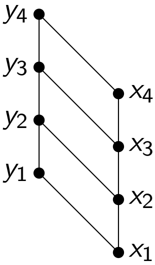

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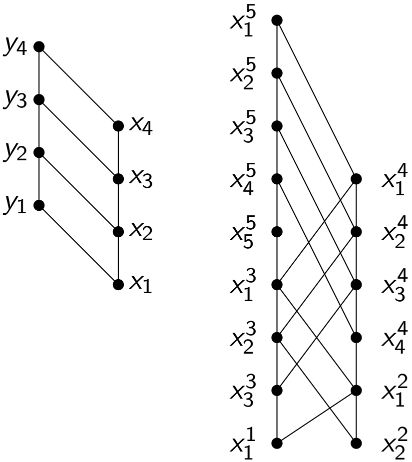

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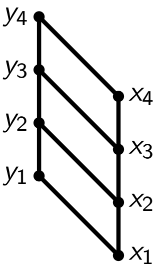

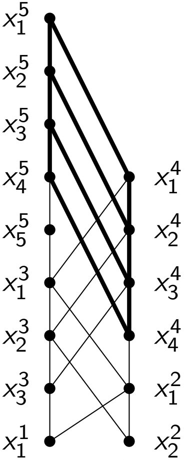

Why Study Ladders?

In the proof of $\mathrm{val}(w) \leq w^{13 \lg w}$, Bosek and Krawczyk found that on-line chain partitioning of general width w posets could be reduced to on-line chain partitioning of L_m-free posets for $1 \leq m \leq 2w^2 + 1$.

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$$\mathrm{val}_{FF}(L_2, w) = w^2$$

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$$w^{lg(m-1)} \leq \mathrm{val}_{FF}(P) \leq w^{2.5 \lg w + 2 \lg m}$$

(Lower bound from Bosek and Matecki)

Upper Bound for L_2-Free, Width w

- Select P with an n-Grundy g coloring so P is minimal; i.e.: for any vertex v, g is not an n-Grundy coloring of $P - v$. Fix $\mathcal{C}$, a Dilworth chain partition of P.

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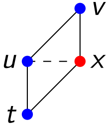

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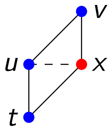
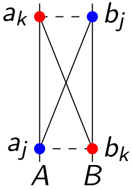


- Any pair of chains in $\mathcal{C}$ share at most two colors.



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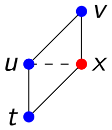

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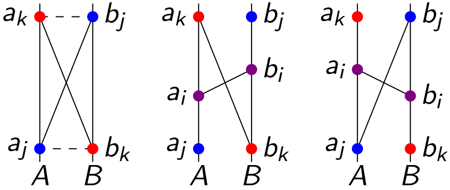

$a_i <_P b_i$ $\qquad$ $b_i <_P a_i$

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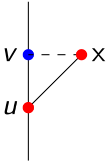

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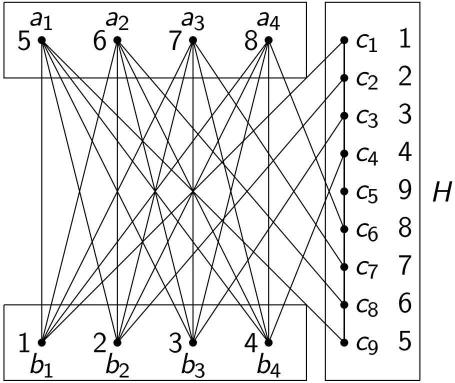
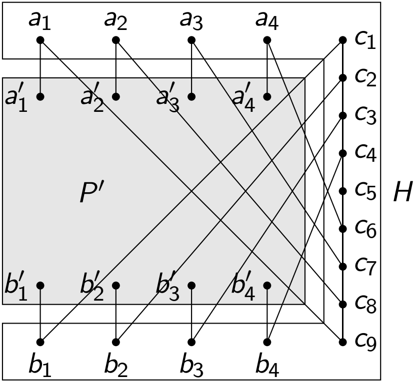


Lower Bound for L_2-Free, Width w

- ... and carefully glue together with an L_2-free poset of width $w-1$ with a $(w-1)^2$-Grundy coloring.



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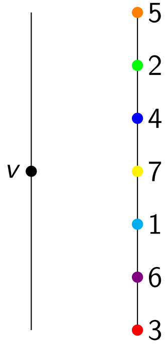

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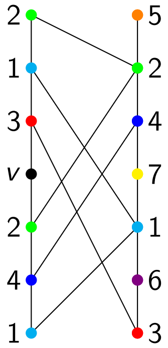

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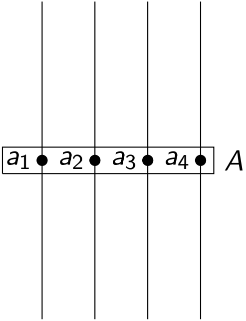
- Select maximum antichain A so that $N := \min_{a \in A} g(a)$ is maximum.



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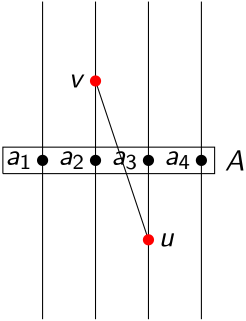
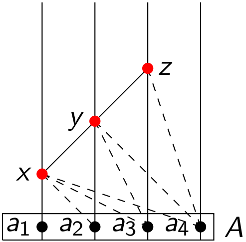


Upper Bound for L_m-Free, Width w

- For each $i \in [N]$ select the "near" witness and "far" witness with property $(*)$ so that they are both on the same side of A.



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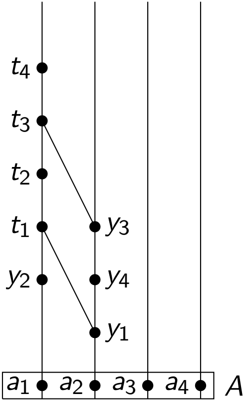

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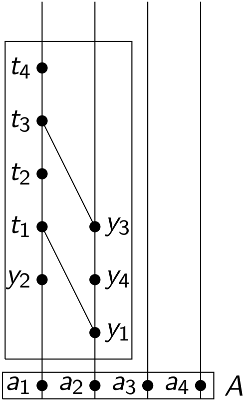

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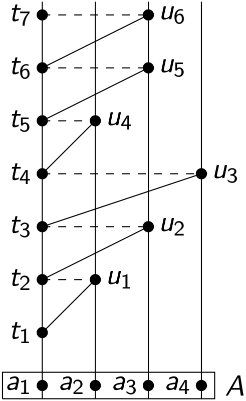
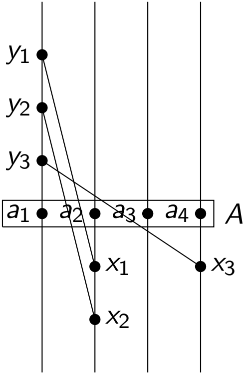


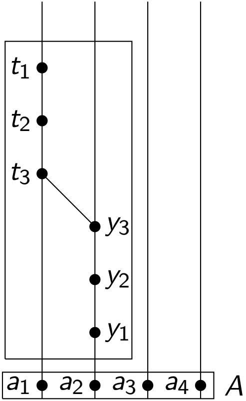
Upper Bound for L_m-Free, Width w

- If the colors of a chain of near witnesses is descending, there can only be m colors in the sequence.



Upper Bound for L_m-Free, Width w

- If a sequence of far witnesses is running "towards" a sequence of near witnesses, this sequence has at most $\mathrm{val}_{\mathrm{FF}}(w/2, L_m)$ colors.



Upper Bound for L_m-Free, Width w

- By E-S, we have at most $m(w-1)^2(w/2)m^2(w-1)^2 \operatorname{val}_{FF}(L_m, w/2)$ far witnesses on each chain in $\mathcal{C}$.

Upper Bound for L_m-Free, Width w

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- $N \leq 2w(w/2)m^2(w-1)^2\,\mathrm{val}_{FF}(L_m, w/2)$

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- $N \leq 2w(w/2)m^2(w-1)^2\,\mathrm{val}_{FF}(L_m, w/2)$
- From our choice of A, colors higher than N for a width $w-1$ poset that is L_m-free.
- $\mathrm{val}_{\mathrm{FF}}(L_m, w) \leq N + \mathrm{val}_{\mathrm{FF}}(L_m, w-1)$.