SIAM Conference on Discrete Mathematics
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Dimension and Height for Posets with Planar Cover Graphs

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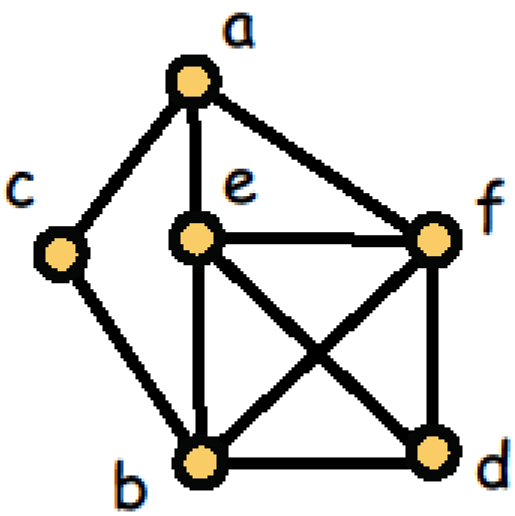
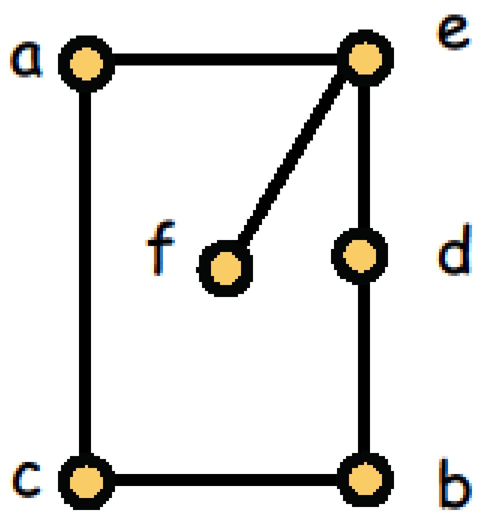
(joint work with W. T. Trotter)

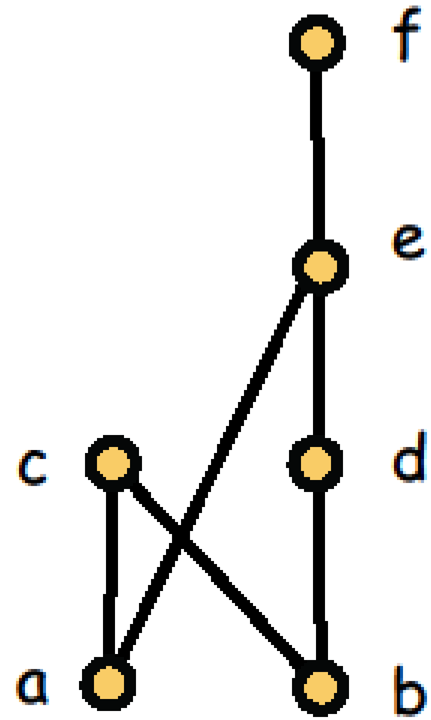
Figures Associated with a Poset



Comparability Graph

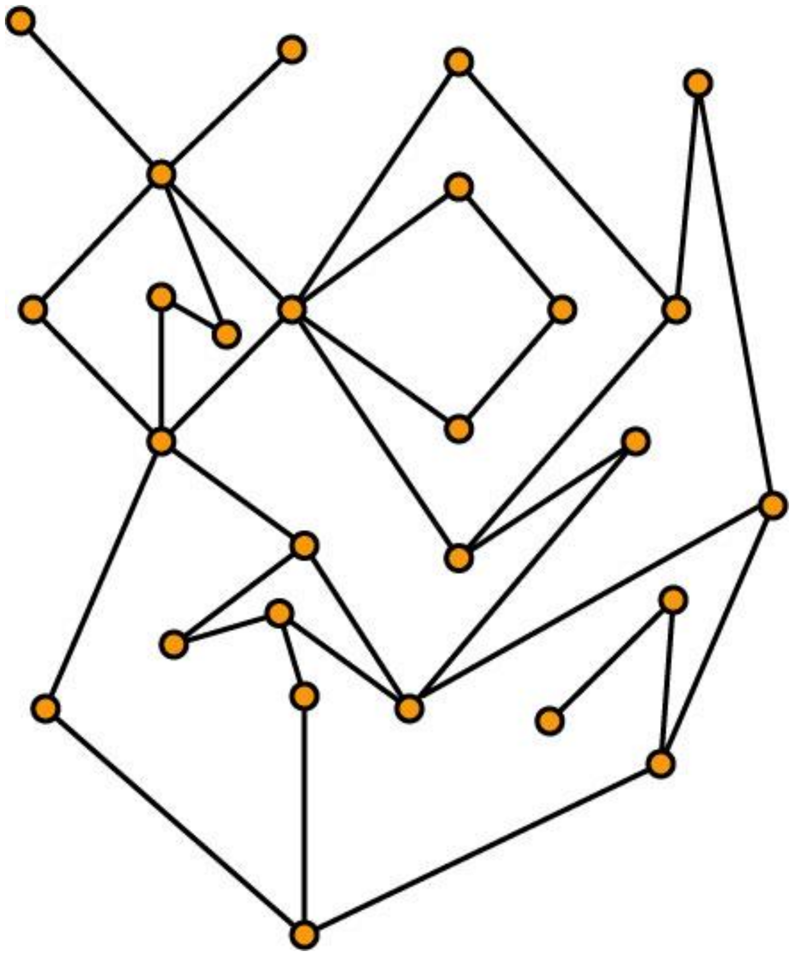


Cover Graph

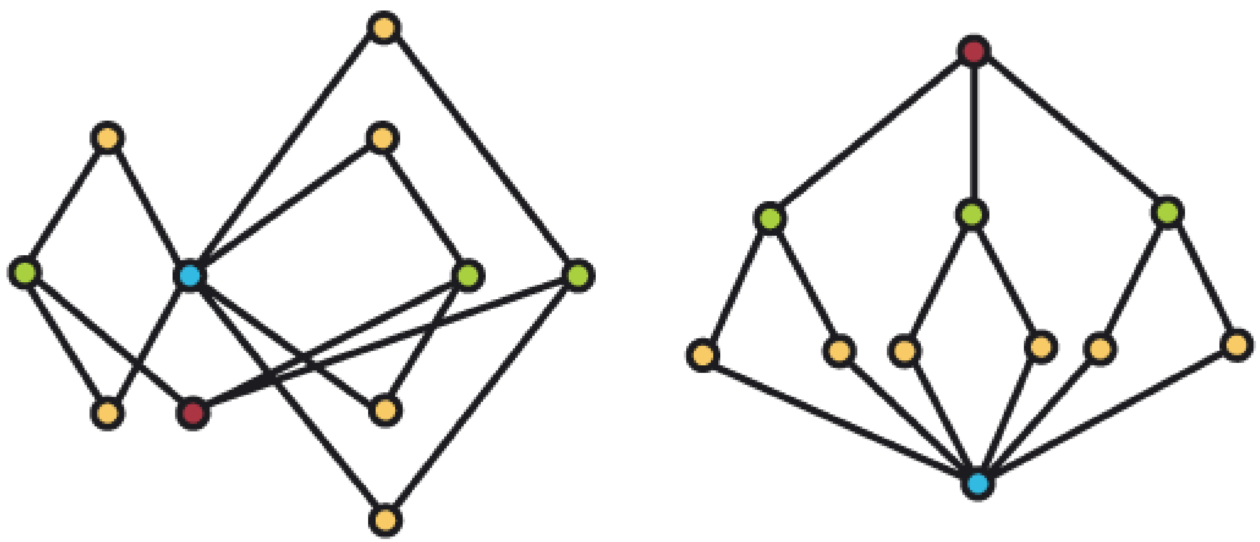


Order Diagram

Planar Poset = Order Diagram is Planar!

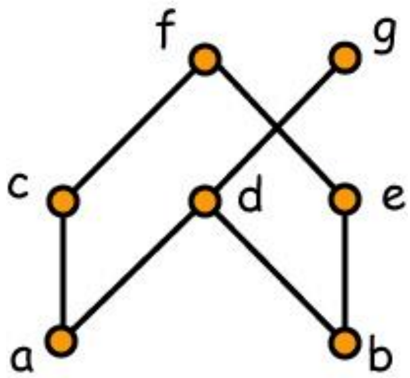


A Non-planar Poset with a Planar Cover Graph



The height 3 non-planar poset on the left has a planar cover graph drawn on the right.

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

$L_2 = a < c < b < d < g < e < f$

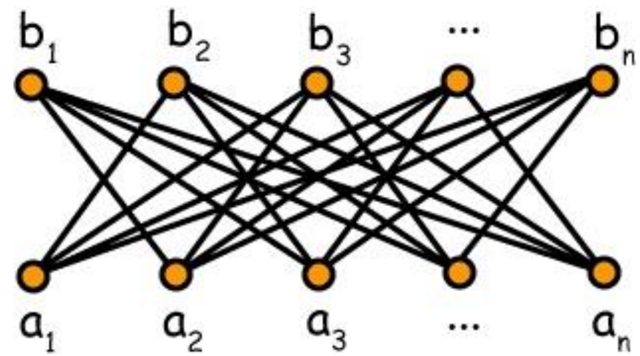
$L_3 = a < c < b < e < f < d < g$

The **dimension** of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$.

Observation Many analogies between dimension and chromatic number.

Standard Examples

S_n



Fact For $n \geq 2$, the **standard example** S_n is a poset of dimension n.

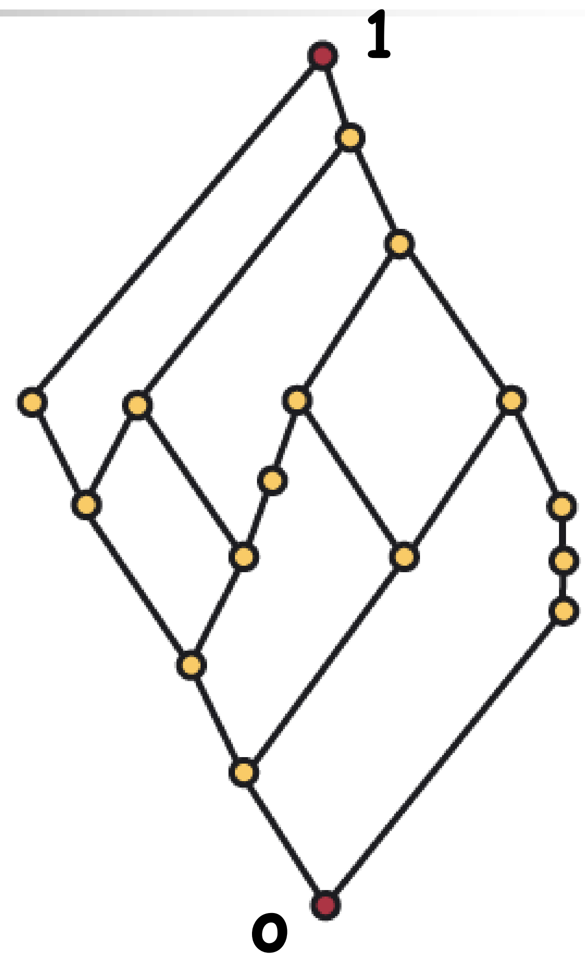
Note If L is a linear extension of S_n, there can only be one value of i for which $a_i > b_i$ in L.

Example $L = a_2 < \ldots < a_n < \mathbf{b_1} < \mathbf{a_1} < b_2 < \ldots < b_n$

Planar Posets with Zero and One

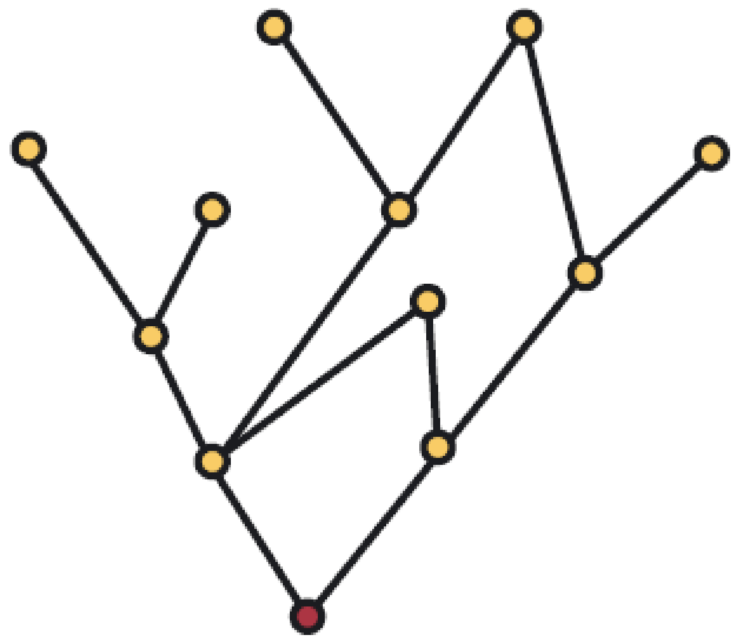
Theorem (Baker, Fishburn and Roberts, '71)

If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.



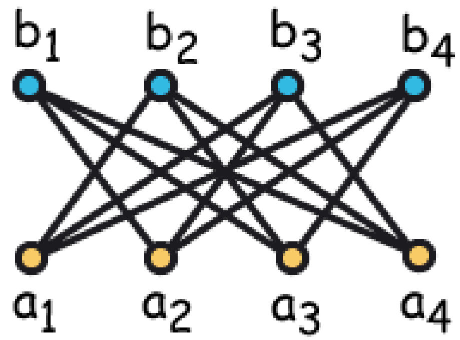
Dimension of Planar Posets

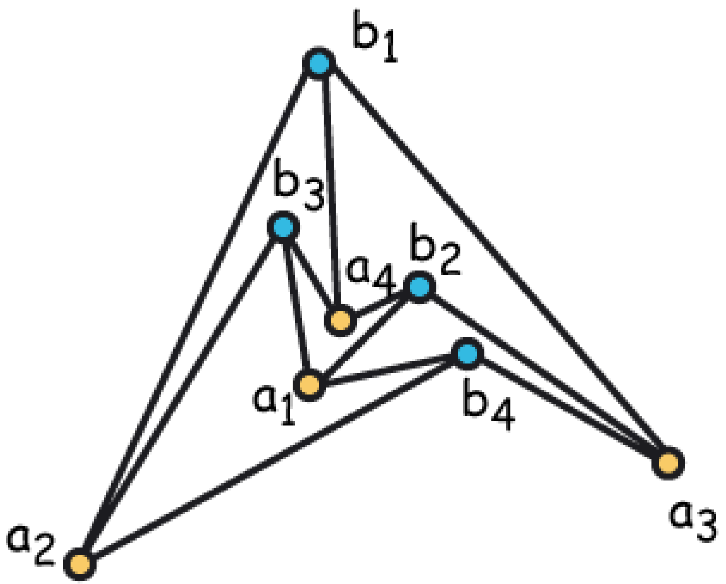
Theorem (Trotter and Moore, '77) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.



A 4-Dimensional Planar Poset

Fact The standard example S_4 is planar!

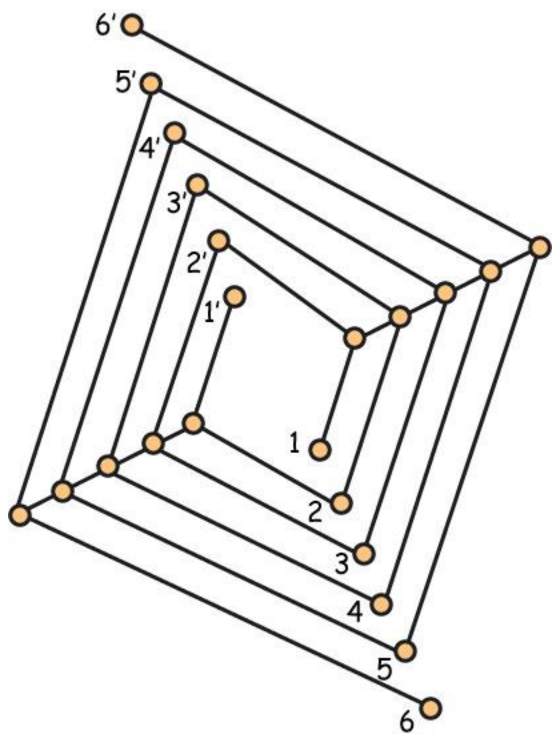




Fact S_n is non-planar for all $n \geq 5$!

No … by Kelly's Construction (1981)

Fact For every $n \geq 5$, the standard example S_n is non-planar, but it is a subposet of a planar poset.



Fact While "dim(P) $\leq$ t" is closed under taking subposets, planarity is not (unlike the situation for graphs).

Schnyder's Theorem

Theorem (Schnyder, '89) A graph is planar if and only if the dimension of its vertex-edge incidence poset is at most 3.

Theorem (Brightwell and Trotter, '97) Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face incidence poset determined by D. Then $\dim(P) \leq 4$.

Planar Cover Graphs, Dimension, and Height 2

Theorem (Felsner, Li and Trotter, '10) If P has height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$.

Fact The inequality is best possible (by S_4).

Planar Cover Graphs, Dimension, and Arbitrary Height

Conjecture (Felsner, Li and Trotter, '09) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation $c_1 = 2$ (antichains) and $c_2 = 4$ (FLT).

Fact Kelly's construction shows that c_h - if it exists - must be at least $h + 1$.

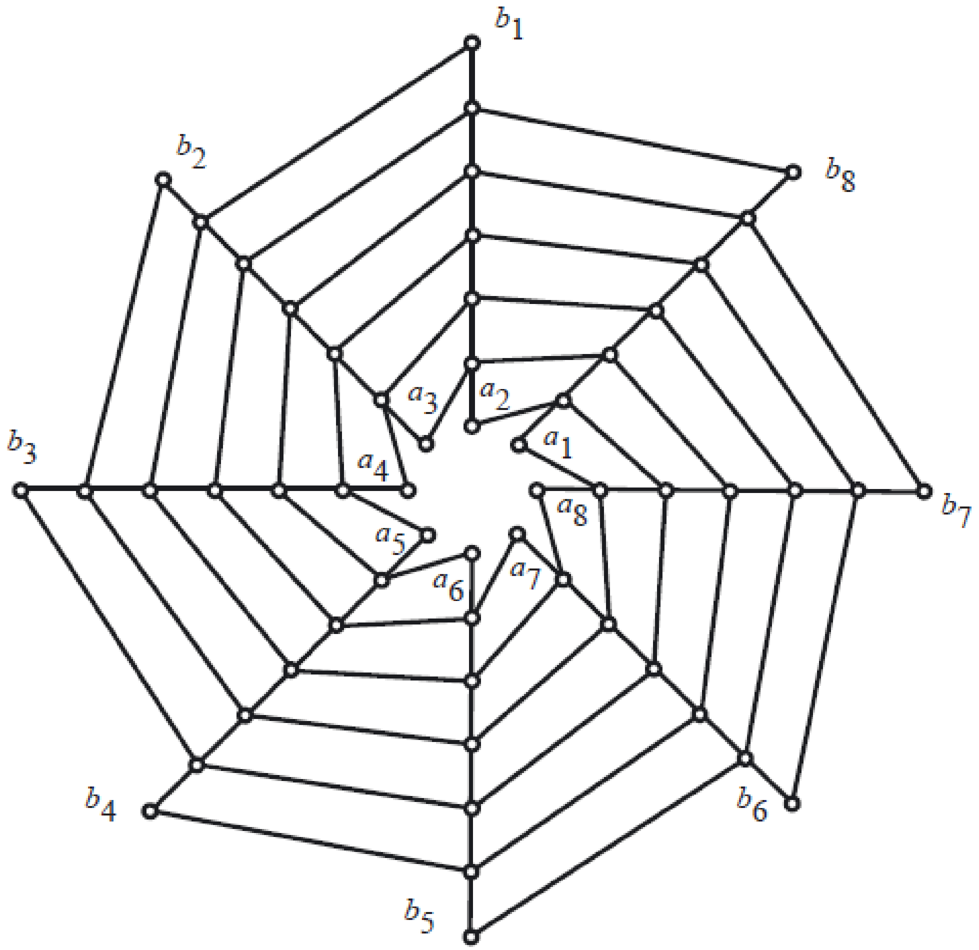
Conjecture Resolved

Theorem (Streib and Trotter, '11) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

However, our argument uses ramsey theory at several key stages … so the constant c_h is **very** large in terms of h.

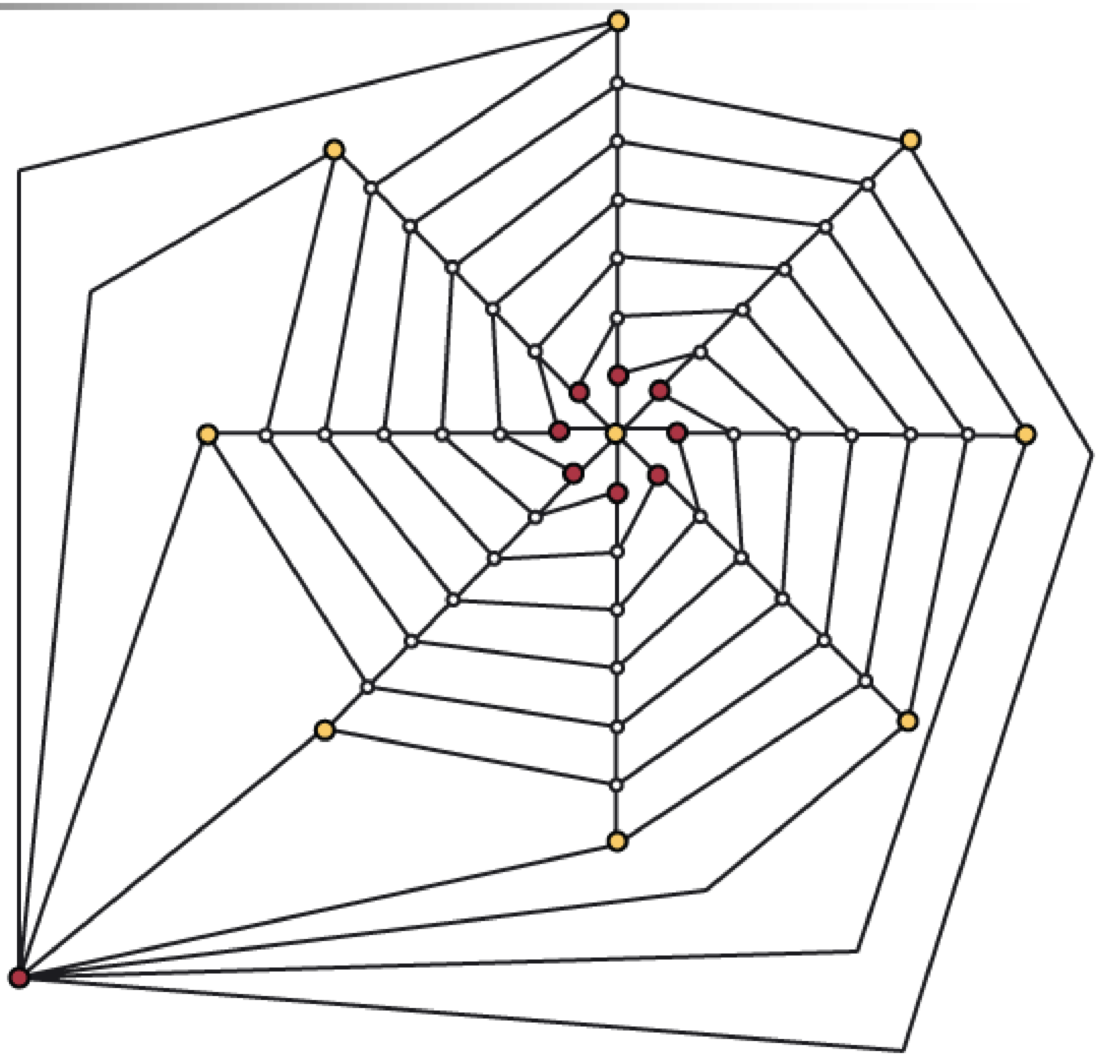
Lower Bound Construction

Fact For every $h \geq 2$, the standard example S_{h+1} is contained in a poset of height h having a planar cover graph.



A Modest Improvement

Fact For every $h \geq 2$, the standard example S_{h+2} is contained in a poset of height h having a planar cover graph.

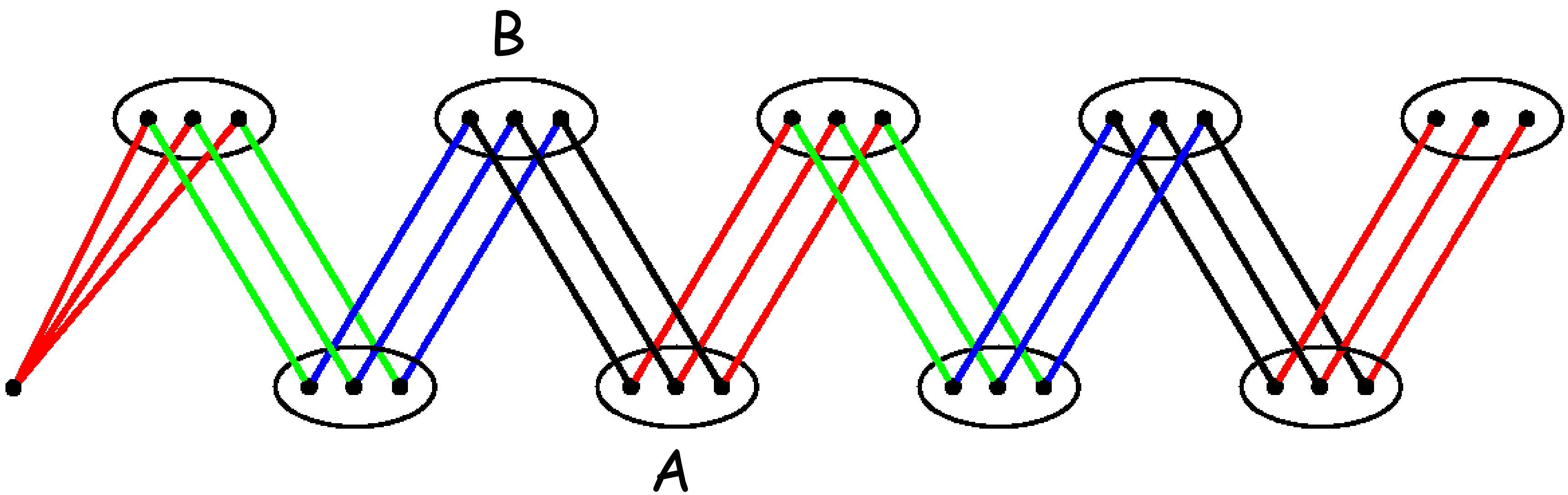


Partitioning Critical Pairs into Reversible Sets

Reduction 1: only consider min/max critical pairs.

Reduction 2: use graph-theoretic contractions and deletions to reduce to a special case: there is a minimal element a_0 under all maximal elements.

$$c_h \leq 4\,c_h^* + 2 + 2$$



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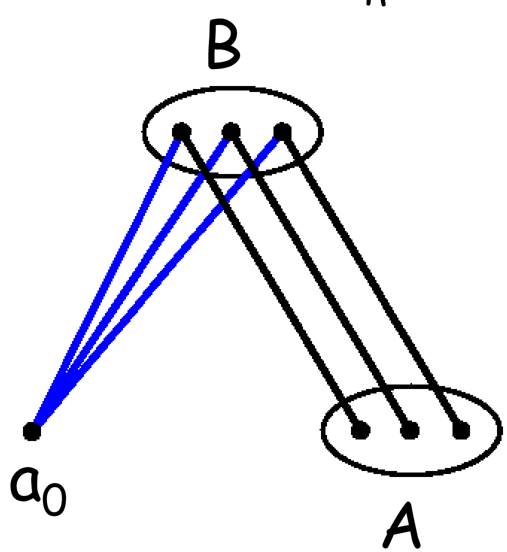
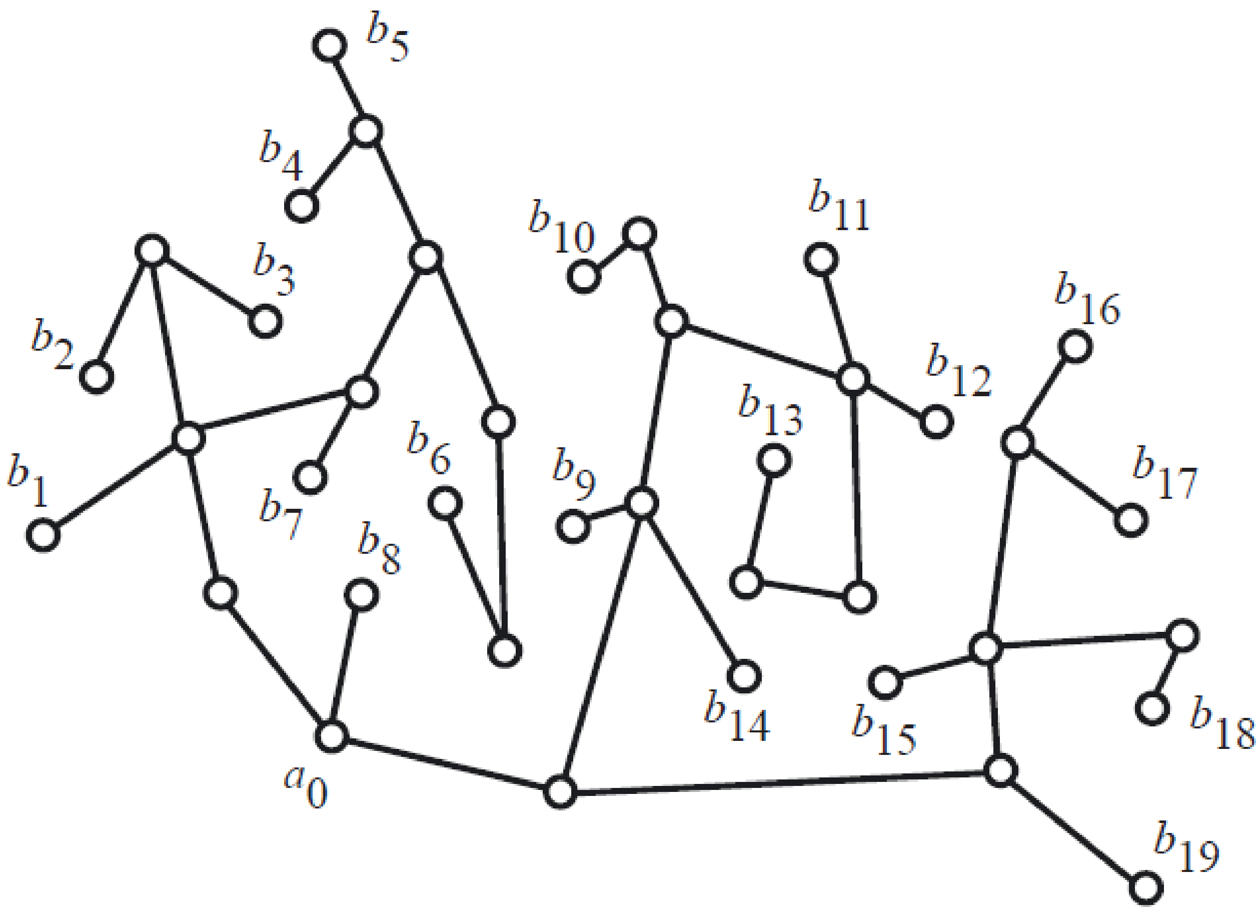


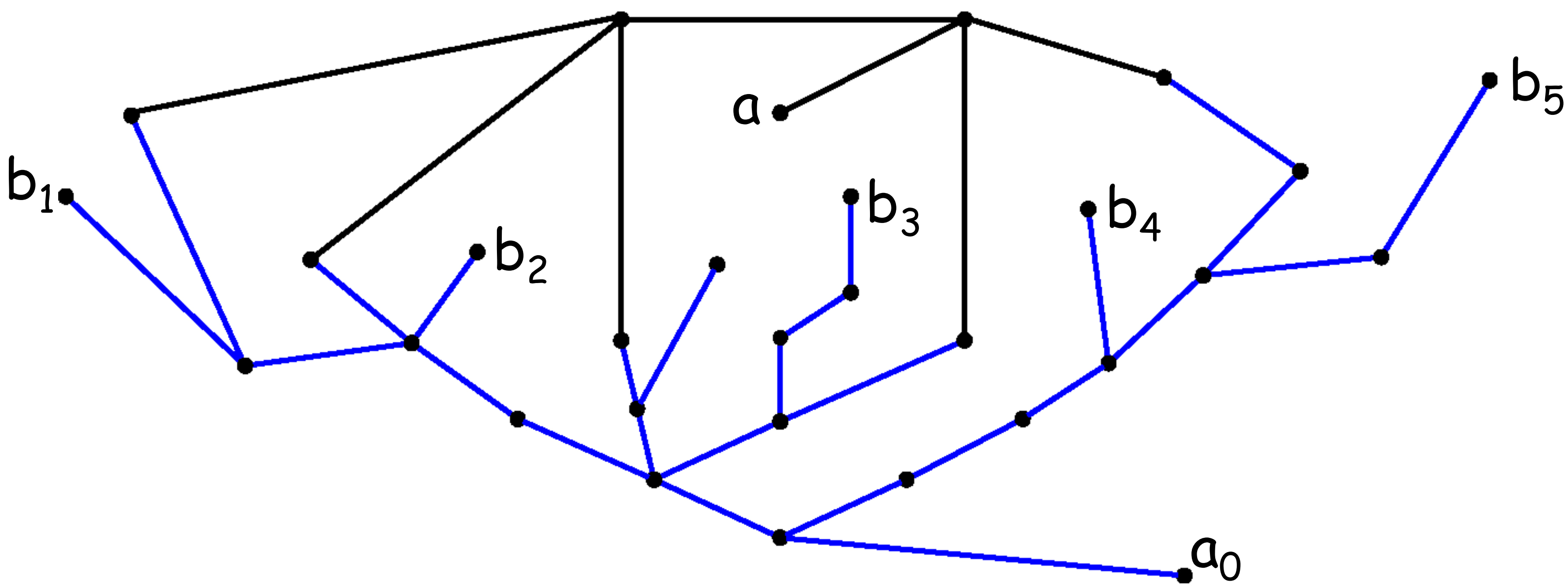
No comparabilities are changed between elements of A and elements of B

An Oriented Tree with a DFS labeling



Regions Determined by the Cover Graph

The **safe** pairs can be reversed in two extensions. For each **dangerous** pair (a,b), we define a **region** in the plane that contains b.



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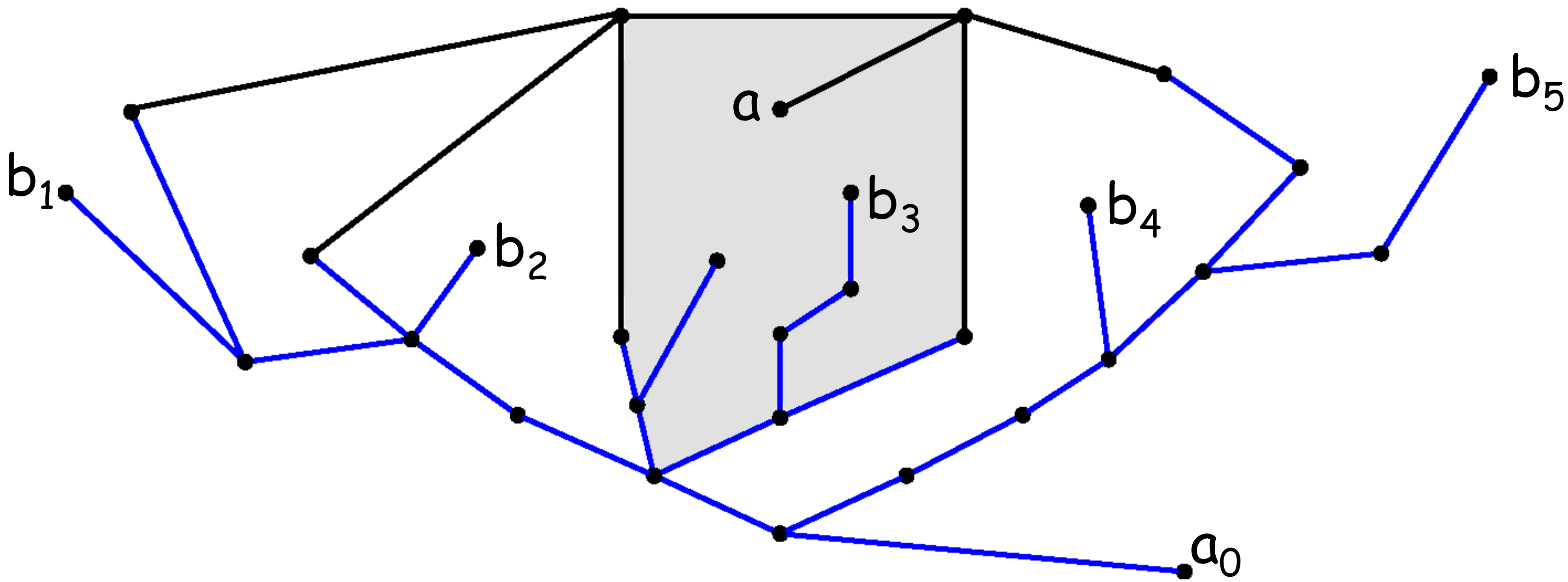
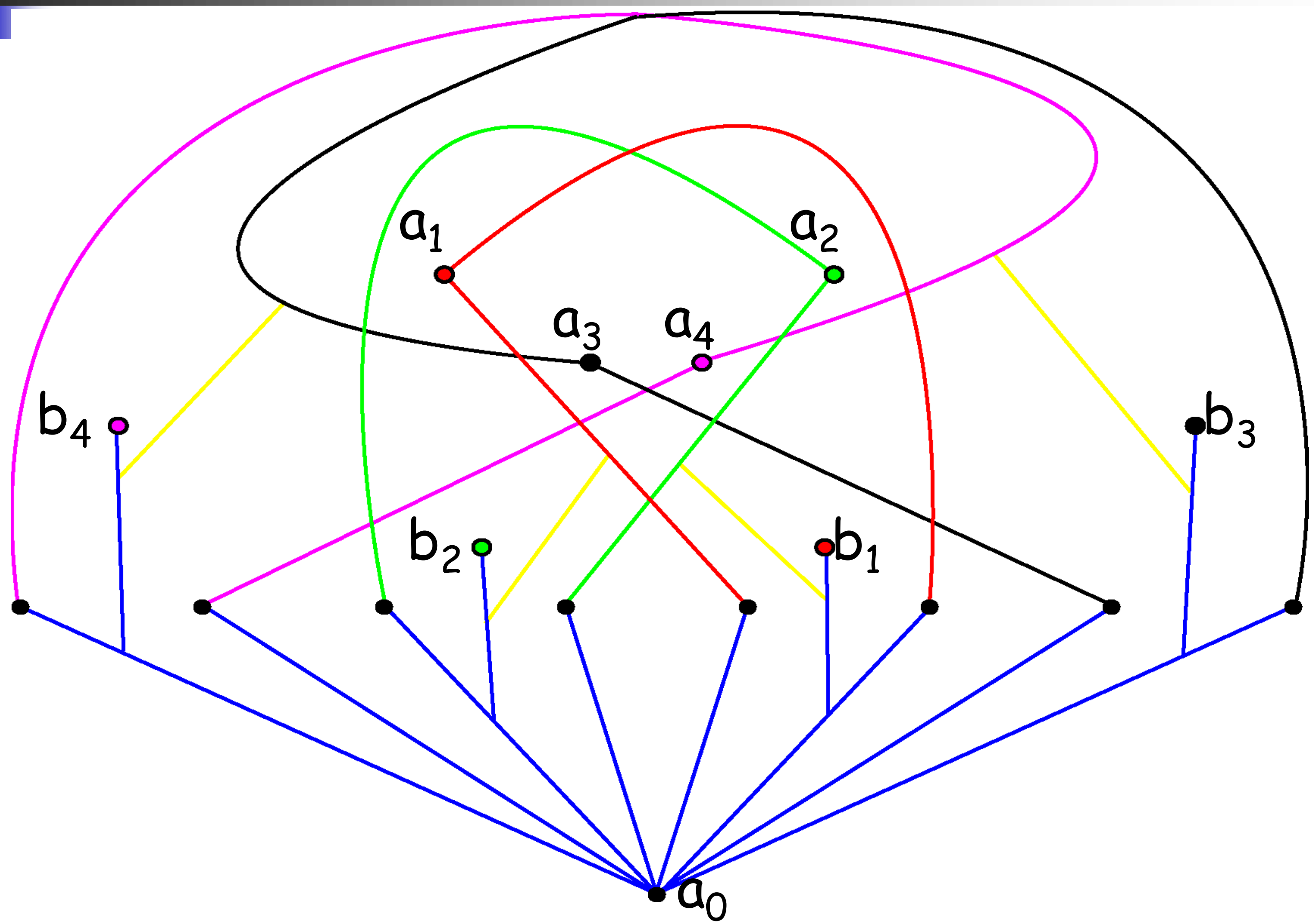


Crossing Regions



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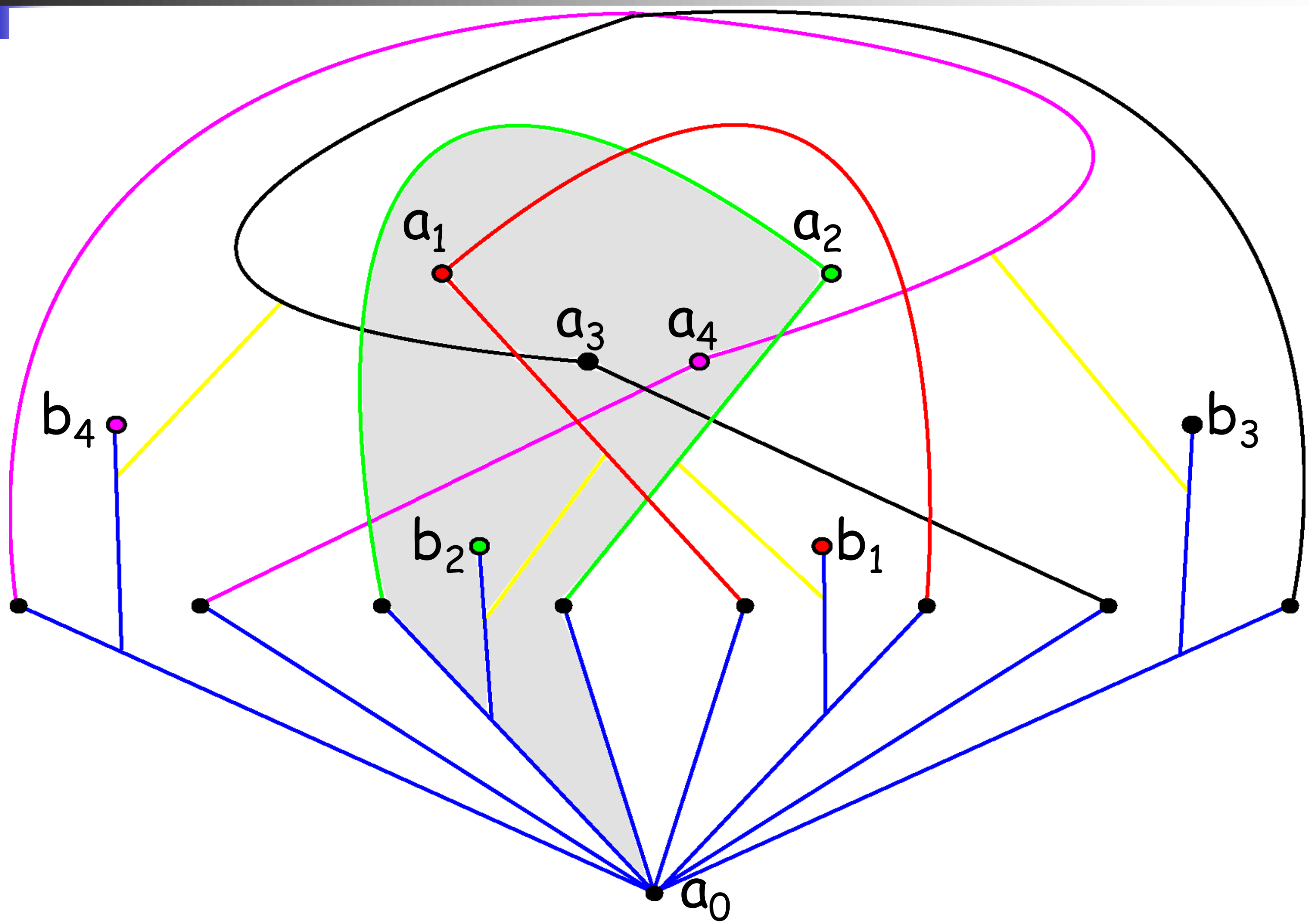
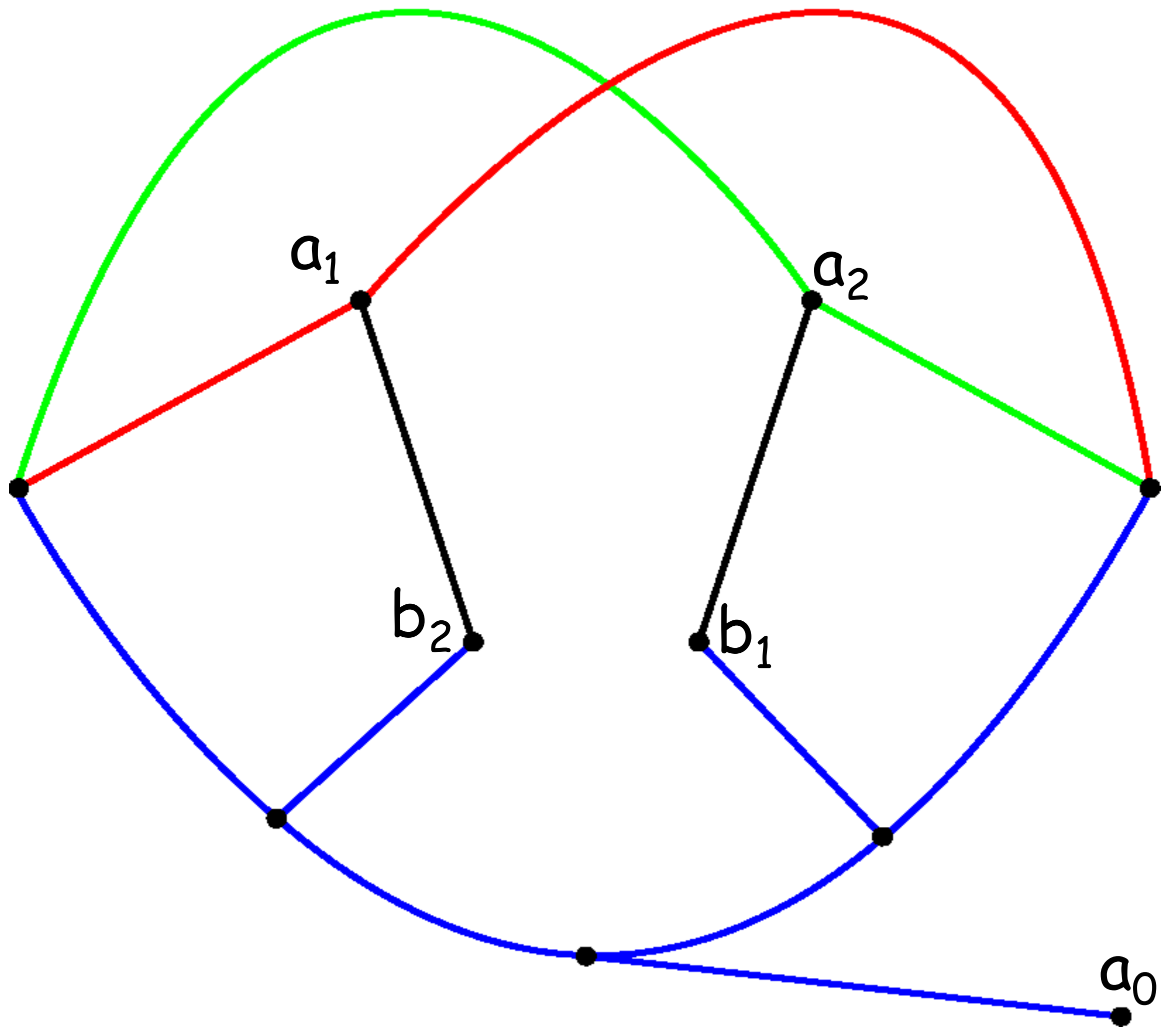
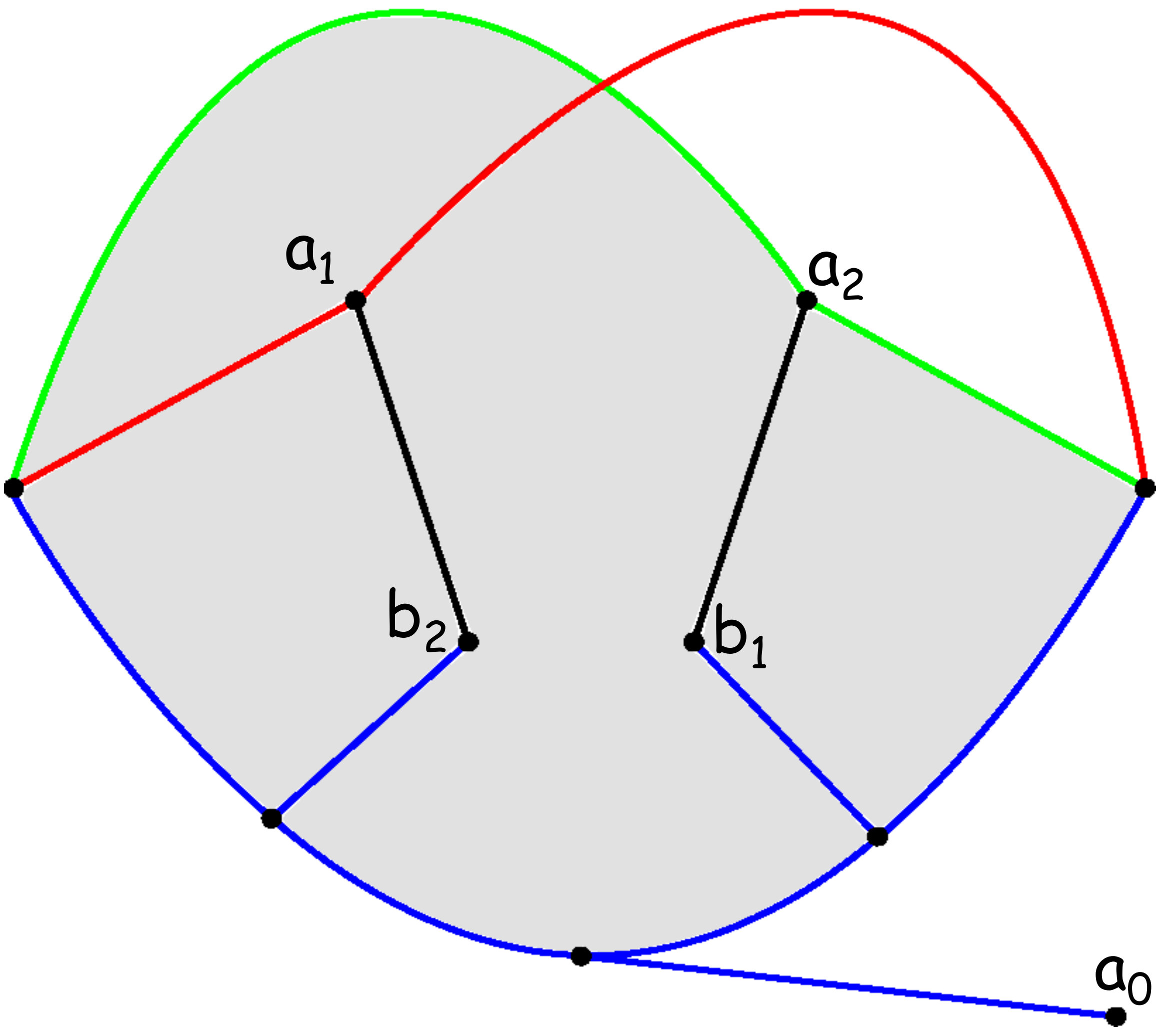


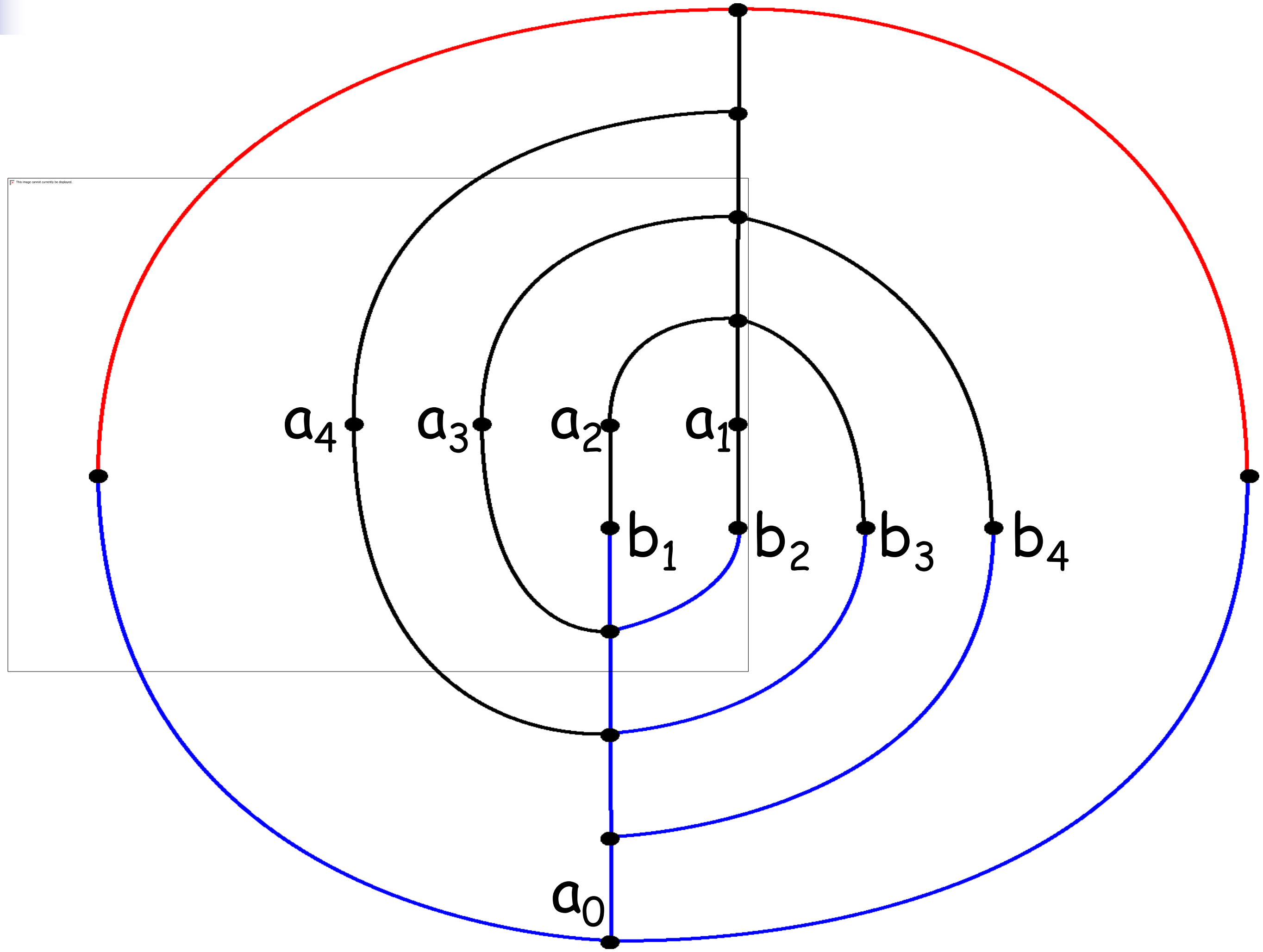
Incomparable Regions



Incomparable Regions



Identical Regions

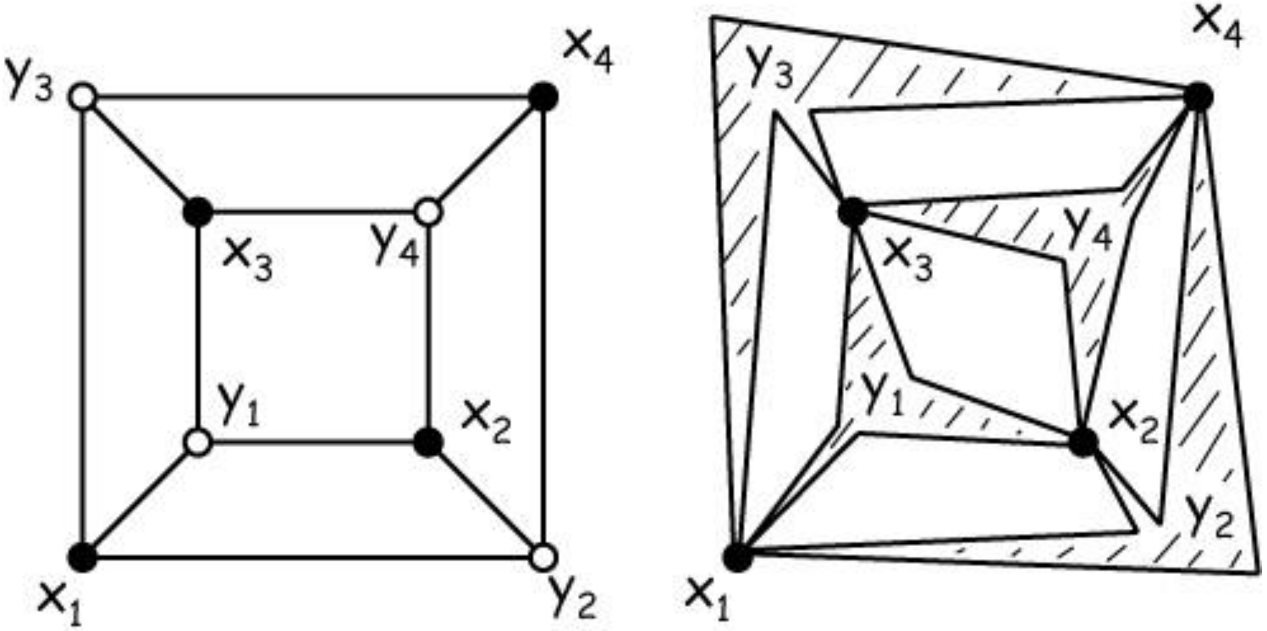


Some Open Questions

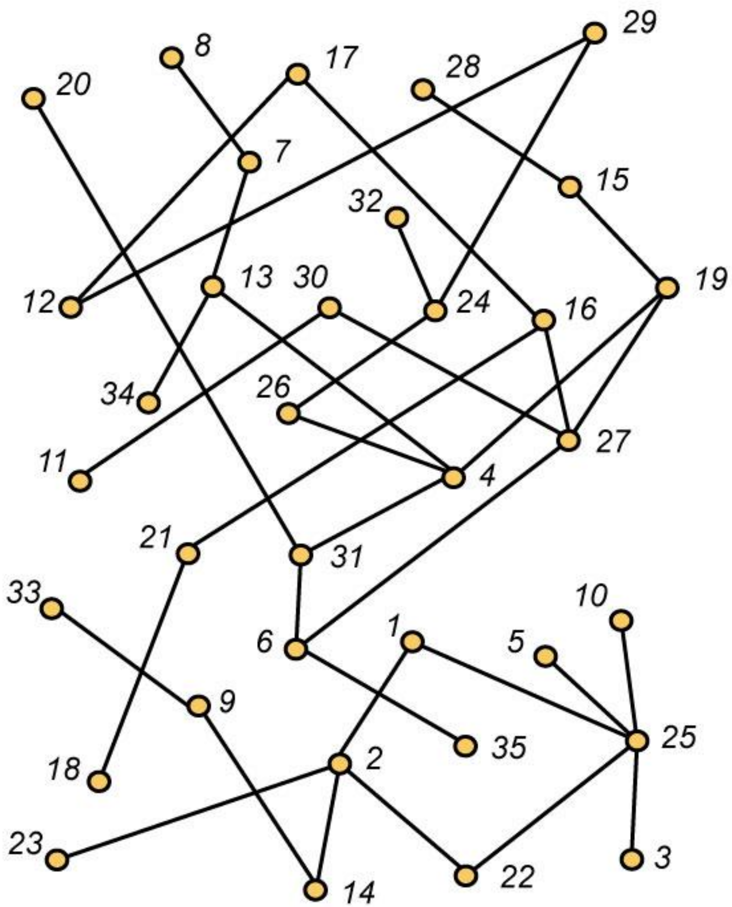
1. Improve the bounds for the constant c_h in the Streib-Trotter theorem.
2. Can we generalize to other surfaces?
3. Which posets are subposets of planar posets? (Recent progress by Cohen and Wiechert)
4. For $t \geq 5$, what is the smallest planar poset having dimension t?
5. For $t \geq 5$, are there planar t-irreducible posets?

Thank you!

Maximal Elements as Faces



Partially Ordered Sets



Proof Highlights

-Use graph-theoretic contractions to reduce to a special case: there is a min a_0 under all maxes

-Find a rooted tree T covering the upset of a_0 and equip it with a DFS labeling scheme

-Give each critical pair a signature of parameters according to its interaction with T

-Prove that these parameters are bounded as a function of h

-Prove that the set of critical pairs with the same signature is reversible

Planar Cover Graphs, Dimension, and Arbitrary Height

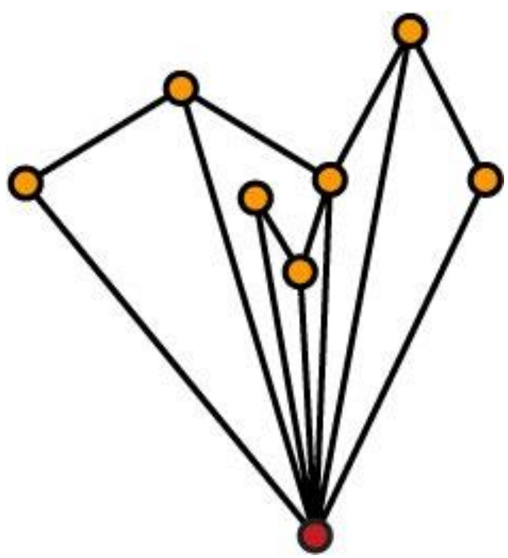
Conjecture (Felsner, Li, Trotter, 2009) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation $c_1 = 2$ (antichains) and $c_2 = 4$ (FLT).

Fact Kelly's construction shows that c_h - if it exists - must be at least $h + 1$.

The Dimension of a Tree

Corollary (Trotter and Moore, 1977) If the diagram of P is a tree, then $\dim(P) \leq 3$.



Planar Multigraphs

Theorem (Brightwell and Trotter, 1993): Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$.

Different drawings may determine posets with different dimensions.