Chromatic Number and Dimension of Incidence Posets

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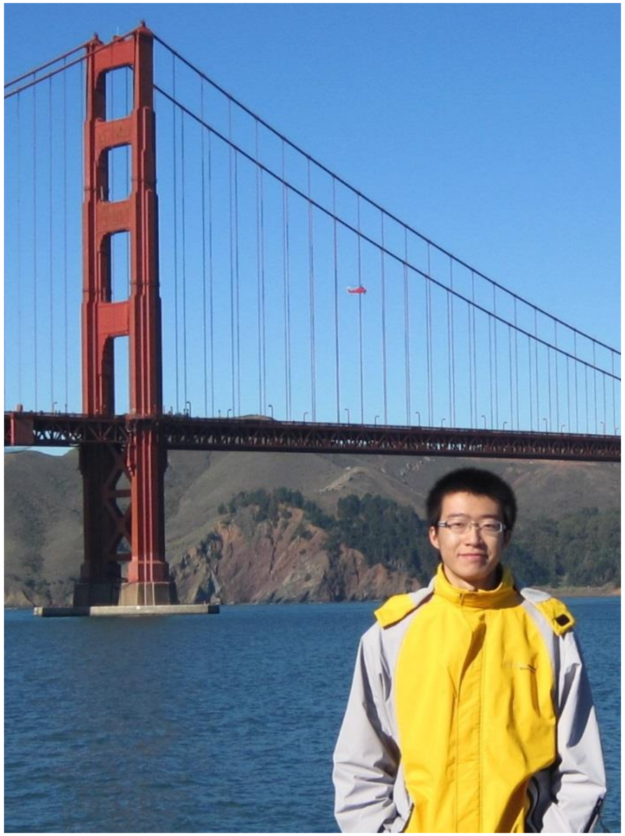
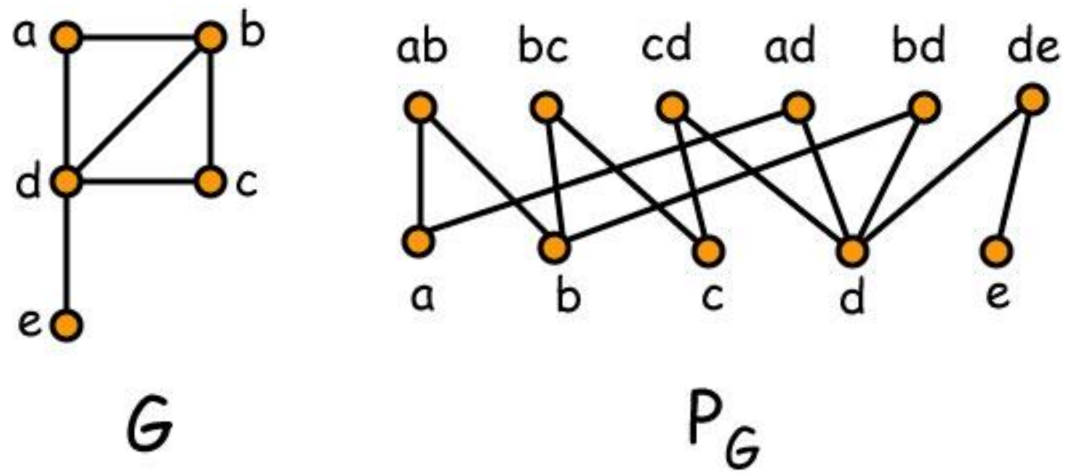
Joint Work with Ruidong Wang



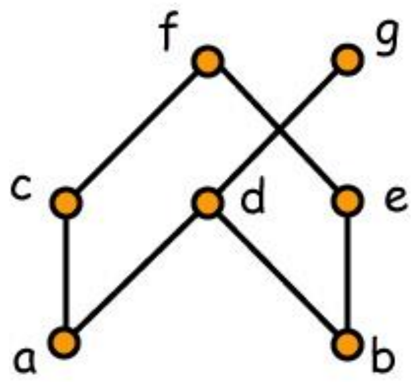
The Incidence Poset of a Graph



The incidence poset of a graph is also called the **vertex-edge** poset of the graph.

Realizers of Posets

A family $\mathbf{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a **realizer** of P if $P = \cap\, \mathbf{F}$, i.e., whenever x is incomparable to y in P, there is some L_i in $\mathbf{F}$ with $x > y$ in L_i.



$L_1 = b < e < a < d < g < c < f$

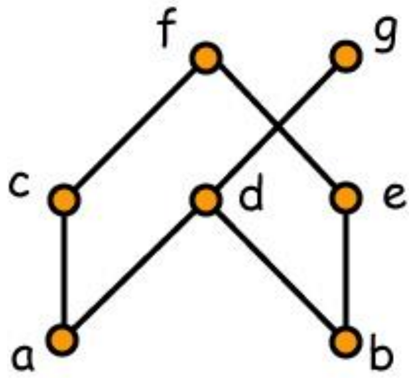
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

$L_4 = b < e < a < c < f < d < g$

$L_5 = a < b < d < g < e < c < f$

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

The **dimension** of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

Schnyder's Theorem

Theorem (Schnyder, 1989) A graph is planar if and only if the dimension of its incidence poset is at most 3.

Remark While the structure developed by Schnyder in proving this theorem remains important today, Barrera-Cruz and Haxell have recently found a very short, direct and elegant proof.

Definition (Barrera-Cruz, Haxell)

Let G be a graph with vertex set V and let P be the incidence poset of G. Then **dim(P)** is the least t for which there are linear orders $L_1, L_2, \ldots, L_t$ on V so that:

1. If x, y, z are distinct vertices and yz is an edge, there is some i for which $x > y$ and $x > z$ in L_i.
2. If x and y are distinct vertices, there is some i for which $x > y$ in L_i.

Two Natural Questions

Let G be a graph and let P be the incidence poset of G.

Question 1 Is the dimension of P bounded as a function of the chromatic number of G?

Question 2 Is the chromatic number of G bounded as a function of the dimension of P?

Yes, for Question 1

Theorem (Agnarsson, Felsner and Trotter, 1999)
If G is a graph with incidence poset P and $\chi(G) = r$, then $\dim(P) = O(\lg \lg r)$.

Remark This result is related to the Erdős-Szekeres theorem for monotonic sequences. Quite accurate asymptotic estimates are known.

Yes, for Question 2 when $\dim(P) \leq 3$

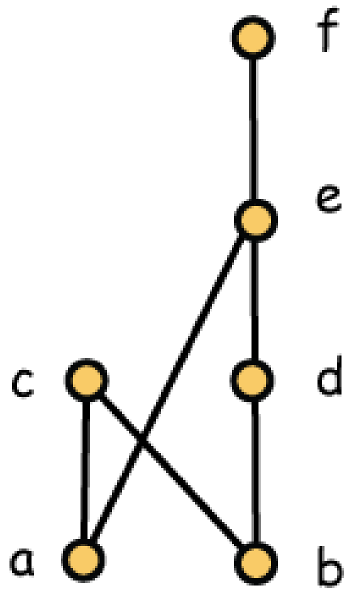
Corollary (Schnyder, 1984) If G is a graph, P is the incidence poset of G and $\dim(P) \leq 3$, then $\chi(G) \leq 4$.

No, for Question 2 when $d \geq 4$

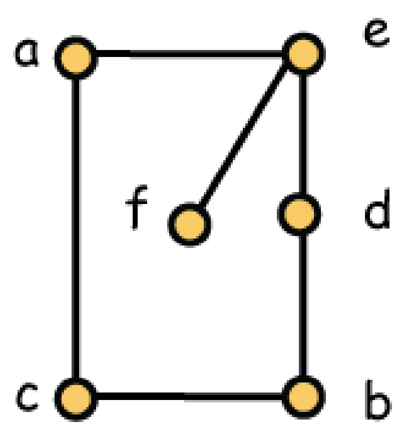
Theorem (Trotter and Wang, 2012) For every $r \geq 1$, there exists a graph G with incidence poset P so that $\dim(P) \leq 4$ and $\chi(G) = r$.

Remark Of course, the inequality $\dim(P) \leq 4$ becomes tight once $r \geq 5$.

Order Diagrams and Cover Graphs



Order Diagram



Cover Graph

Classic Results (1)

Theorem (B. Descartes, 1943) For every $r \geq 1$, there exists a poset P with cover graph G so that height(P) = r and $\chi(G) = r$.

Theorem (Nešetřil and Rődl, 1983) For every $r, g \geq 1$, there exists a poset P with cover graph G so that height(P) = r, girth(G) $\geq g$ and $\chi(G) = r$.

Theorem (Felsner and Trotter, 1995) For every $r \geq 1$, there exists an interval order P with cover graph G so that $\chi(G) = r$. The height of P must be exponentially large in r.

Classic Results (2)

Theorem (Bollobás, 1977) For every $r \geq 1$, there exists a lattice L with cover graph G so that $\chi(G) = r$.

Theorem (Kříž and Nešetřil, 1991) For every $r \geq 1$, there exists a poset P with cover graph G so that $\dim(P) = 2$ and $\chi(G) = r$.

An Application

Let $r \geq 1$, and let P be a poset with cover graph G so that $\dim(P) = 2$ and $X(G) = r$. Also, let $\{L_1, L_2\}$ be a realizer for P.

For each $i = 1, 2$, let M_i be the dual of L_i. Also, let Q be the incidence poset of G. Then, using the Barrera-Cruz/Haxell definition, the family $\{L_1, L_2, M_1, M_2\}$ witnesses that $\dim(Q) \leq 4$.

A Graph Parameter (Kříž and Nešetřil)

Let G be a graph with vertex set V. Set **eye(G)** to be the least t for which there are linear orders $L_1, L_2, \ldots, L_t$ on V so that:

1. If x, y, z are distinct vertices and yz is an edge in G, there is some i for which x is not between y and z in L_i

Classic Results (Kříž and Nešetřil)

Corollary For every $r \geq 1$, there exists a graph G so that $\mathrm{eye}(G) \leq 2$ and $\chi(G) = r$.

Corollary For every $g, r \geq 1$, there exists a graph G so that $\mathrm{eye}(G) \leq 3$, $\mathrm{girth}(G) \geq g$ and $\chi(G) = r$.

Question Is it true that for every $g, r \geq 1$, there exists a graph G so that $\mathrm{eye}(G) \leq 2$, $\mathrm{girth}(G) \geq g$ and $\chi(G) = r$?

Answer and New Question

Theorem (Trotter and Wang, 2012) For every $g, r \geq 1$, there exists a graph G so that $\text{eye}(G) \leq 2$, $\text{girth}(G) \geq g$ and $\chi(G) = r$.

Question Is it true that for every $g, r \geq 1$, there exists a poset P with cover graph G so that $\dim(P) \leq 2$, $\text{girth}(G) \geq g$ and $\chi(G) \geq r$?

Remark We expect that the answer is no!!