An extremal problem on crossing vectors

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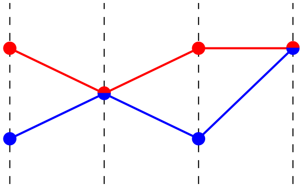
problem formulation

a, b — vectors in $\mathbb{Z}^w$

$a \leqslant b$ if $a_i \leqslant b_i$ for all coordinates $i = 1, \ldots, w$



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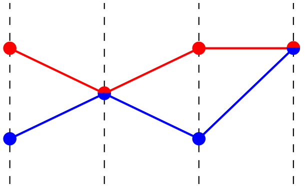
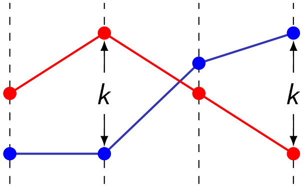
a, b are k-crossing if there are two coordinates i, j such that $a_i - b_i \geqslant k$ and $b_j - a_j \geqslant k$





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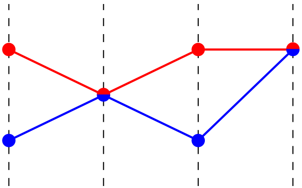

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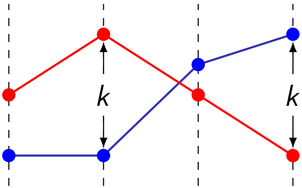

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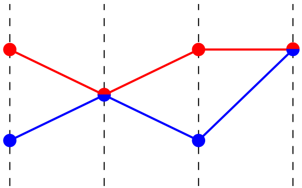

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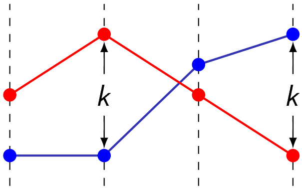

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$f(k, w) = ?$

motivation

A, B — maximum antichains in a poset P

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$g(k,w)$ = the maximum width of the lattice of maximum antichains in a $(k+k)$-free poset of width w

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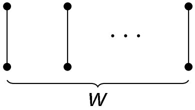

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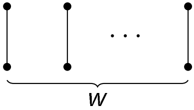

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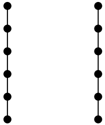

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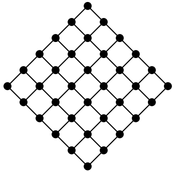

reduction

Theorem

$g(k+1, w) \leqslant f(k, w)$

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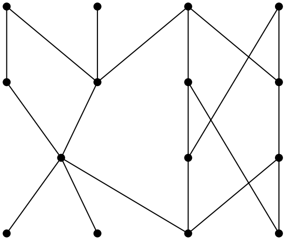

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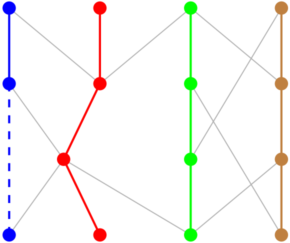

Dilworth partition into w chains

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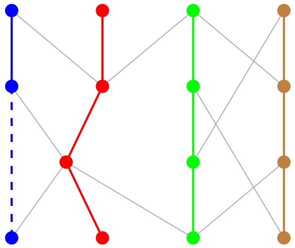

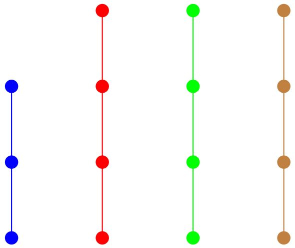

Dilworth partition into w chains $\rightarrow$ w coordinates

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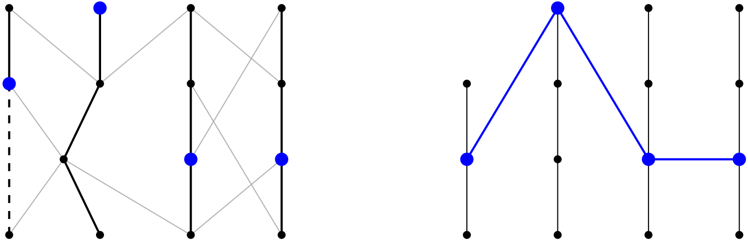

Dilworth partition into w chains $\rightarrow$ w coordinates

maximum antichain $\rightarrow$ vector

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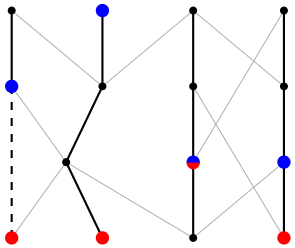

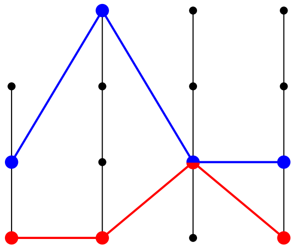

Dilworth partition into w chains	$\to$	w coordinates
maximum antichain	$\to$	vector
comparable antichains	$\leftrightarrow$	comparable vectors

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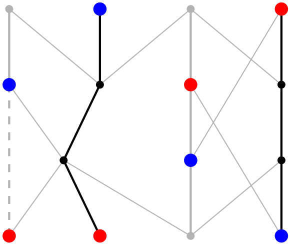

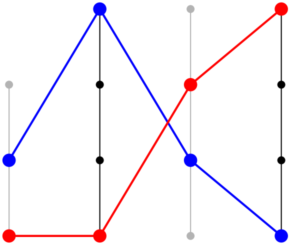

Dilworth partition into w chains	$\rightarrow$	w coordinates
maximum antichain	$\rightarrow$	vector
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$(k+1)+(k+1)$	$\leftarrow$	k-crossing vectors

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Theorem

$f(k,w) \leqslant k^w - (k-1)^w$

$f(k,w) \leqslant k^w - k^2(k-1)^{w-2} \quad (w \geqslant 2)$

$f(k,w) \leqslant wk^{w-1}/3$

lower bound

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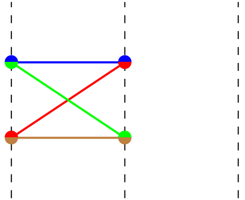
$$\{(a_1, \ldots, a_w) \in \{0, \ldots, k-1\}^{w-1} \times \mathbb{Z} : a_1 + \ldots + a_w = 0\}$$



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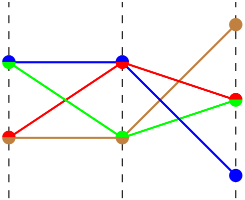

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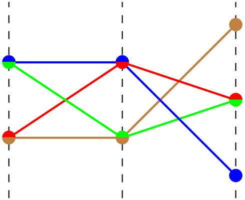

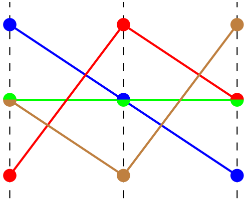

other completely different examples

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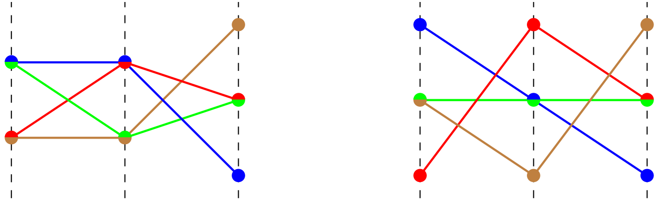

other completely different examples

Conjecture (Felsner, Krawczyk, Micek)

$f(k, w) = k^{w-1}$

exact bounds

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- $f(k, 1) = 1$ (trivial)
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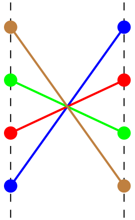
- $f(k, 2) = k$



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- $f(k,3) = k^2$ (to be presented)
- $f(k,w) = ?$ $(k \geqslant 2,\ w \geqslant 4)$

case $w = 3$

Theorem

$f(k, 3) = k^2$

$\mathcal{F}$ — family of pairwise incomparable non-k-crossing vectors in $\mathbb{Z}^3$
minimizing $\max\{|a_3 - b_3| : a, b \in \mathcal{F}\}$

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levels: $\mathcal{F}_i = \{a \in \mathcal{F} : a_3 = i\}$

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$a \to b$ if increasing a_3 by 1 yields:

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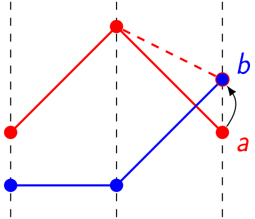

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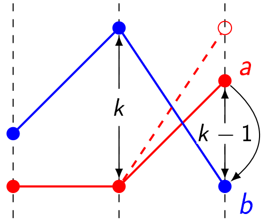

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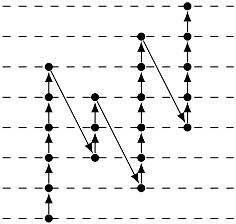

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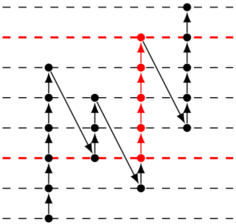

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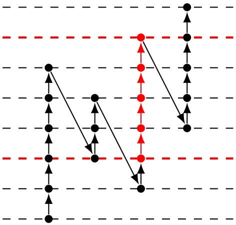

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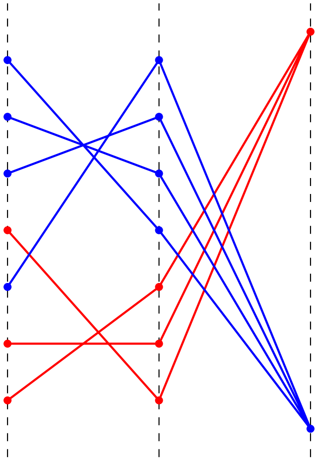
3. $\sum_{i \equiv r \pmod{k}} |\mathcal{F}_i| \leqslant k$

case $w = 3$ (continued)

Claim

$|\mathcal{F}_i| + |\mathcal{F}_{i+k}| \leqslant k$



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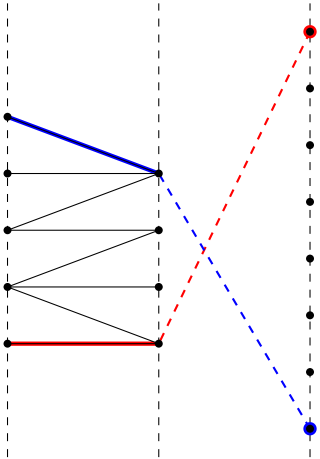

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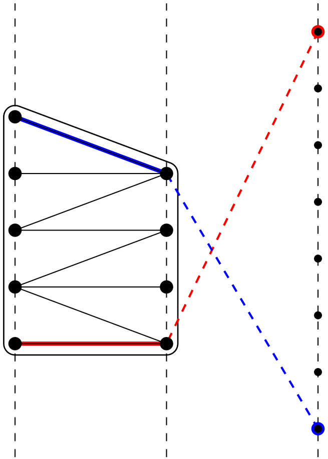

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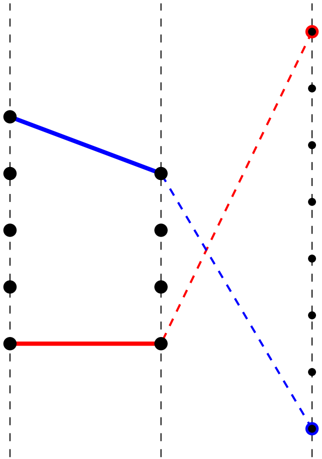

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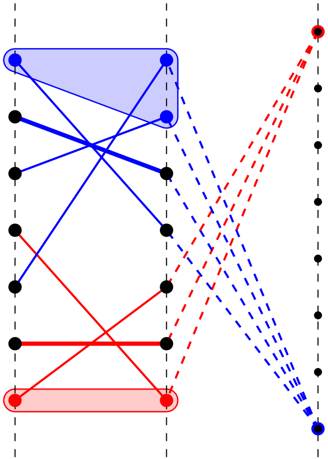
$\#\text{black} + \#\text{blue} + \#\text{red} \leqslant 2k$



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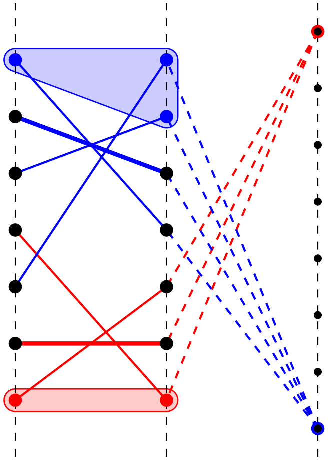
$|\mathcal{F}_i| + |\mathcal{F}_{i+k}| =$
$\#\text{blue} + \#\text{red} + 2 \leqslant k$



summary

$f(k,w)$ = the maximum size of a family of pairwise incomparable and non-k-crossing vectors in $\mathbb{Z}^w$

$g(k,w)$ = the maximum width of the lattice of maximum antichains in a $(k+k)$-free poset of width w

Theorem

$$k^{w-1} \leqslant g(k+1,w) \leqslant f(k,w) \leqslant \begin{cases} k^w - (k-1)^w \\ k^w - k^2(k-1)^{w-2} \quad (w \geqslant 2) \\ wk^{w-1}/3 \end{cases}$$

Conjecture

$$g(k+1,w) = f(k,w) = k^{w-1}$$

true for $k = 1$ or $1 \leqslant w \leqslant 3$

Theorem

$$g(3,w) = f(2,w)$$