Linear Discrepancy of Posets

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

Joint work with
Jeong-Ok Choi and Kevin G. Milans

The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but "more urgent" is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but "more urgent" is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

Def. A linear extension L of P is an order-preserving bijection $L\colon P \to [n]$. An extension L is k-tight if $|L(x) - L(y)| \le k$ whenever $x \| y$. The linear discrepancy $\mathrm{ld}(P)$ is $\min\{k\colon P$ has a k-tight linear extension$\}$.

The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but "more urgent" is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

Def. A linear extension L of P is an order-preserving bijection $L\colon P \to [n]$. An extension L is k-tight if $|L(x) - L(y)| \le k$ whenever $x \| y$. The linear discrepancy $\mathrm{ld}(P)$ is $\min\{k\colon P$ has a k-tight linear extension$\}$.

When P is a chain, $\mathrm{ld}(P) = 0$.

The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but “more urgent” is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

Def. A linear extension L of P is an order-preserving bijection $L \colon P \to [n]$. An extension L is k-tight if $|L(x) - L(y)| \le k$ whenever $x \| y$. The linear discrepancy $\mathrm{ld}(P)$ is $\min\{k \colon P$ has a k-tight linear extension$\}$.

When P is a chain, $\mathrm{ld}(P) = 0$.

When P is an antichain, $\mathrm{ld}(P) = |P| - 1$.

The Problem

Tanenbaum–Trenk–Fishburn [2001]: Patients must be seen in a linear order, but "more urgent" is a poset P. We must treat x before y if $x < y$ in P. If $x \| y$ in P, then x and y should be treated not long apart.

Def. A linear extension L of P is an order-preserving bijection $L \colon P \to [n]$. An extension L is k-tight if $|L(x) - L(y)| \le k$ whenever $x \| y$. The linear discrepancy $\mathrm{ld}(P)$ is $\min\{k \colon P$ has a k-tight linear extension$\}$.

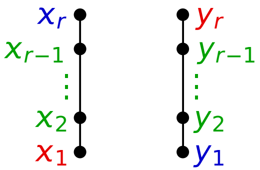
When P is a chain, $\mathrm{ld}(P) = 0$.

When P is an antichain, $\mathrm{ld}(P) = |P| - 1$.

More generally, $\mathrm{ld}(P) \ge \mathrm{width}(P) - 1$.

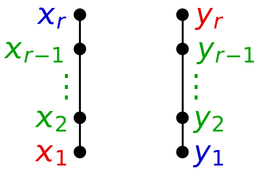
An example

Ex. $\mathbf{r}+\mathbf{r}$; two disjoint r-element chains.



An example

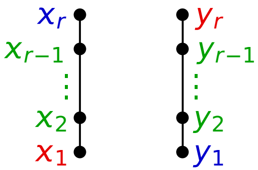
Ex. $\mathbf{r} + \mathbf{r}$; two disjoint r-element chains.



To get tightness below $2r - 1$, top and bottom of extension must come from same chain. For the other chain, want to get bottom high and top low.

An example

Ex. $\mathbf{r}+\mathbf{r}$; two disjoint r-element chains.

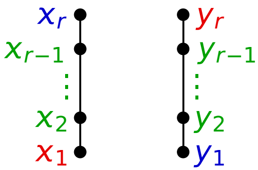


To get tightness below $2r-1$, top and bottom of extension must come from same chain. For the other chain, want to get bottom high and top low.

$$L = x_1, x_2, \dots, \underline{y_1, y_2, \dots, y_{r-1}, y_r}, \dots, x_{r-1}, x_r$$

An example

Ex. $\mathbf{r}+\mathbf{r}$; two disjoint r-element chains.



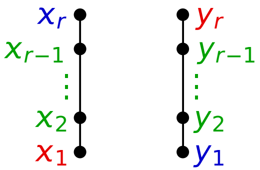
To get tightness below $2r-1$, top and bottom of extension must come from same chain. For the other chain, want to get bottom high and top low.

$$L = x_1, x_2, \ldots, \underline{y_1, y_2, \ldots, y_{r-1}, y_r}, \ldots, x_{r-1}, x_r$$

Thus $\mathrm{ld}(\mathbf{r}+\mathbf{r}) = \left\lfloor \frac{3r-1}{2} \right\rfloor$.

An example

Ex. $\mathbf{r}+\mathbf{r}$; two disjoint r-element chains.



To get tightness below $2r-1$, top and bottom of extension must come from same chain. For the other chain, want to get bottom high and top low.

$$L = x_1, x_2, \ldots, \underline{y_1, y_2, \ldots, y_{r-1}, y_r}, \ldots, x_{r-1}, x_r$$

Thus $\operatorname{ld}(\mathbf{r}+\mathbf{r}) = \left\lfloor \frac{3r-1}{2} \right\rfloor$.

• In this example, each element is incomparable to exactly r other elements.

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Prop. (Rautenbach [2005]) $\mathrm{ld}(P) \le 2r - 1$.

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Prop. (Rautenbach [2005]) $\mathrm{ld}(P) \le 2r-1$.

Pf. Every linear extension L is $(2r-1)$-tight. If $x \| y$, then between them are at most $r-1$ elements incomp. to x $+$ at most $r-1$ incomp. to y, so $|L(x)-L(y)| \le 2r-1$. ∎

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Prop. (Rautenbach [2005]) $\mathrm{ld}(P) \le 2r - 1$.

Pf. Every linear extension L is $(2r-1)$-tight. If $x \| y$, then between them are at most $r-1$ elements incomp. to x $+$ at most $r-1$ incomp. to y, so $|L(x) - L(y)| \le 2r-1$. ■

Conj. (Tanenbaum-Trenk-Fishburn [2001])
Always $\mathrm{ld}(P) \le \left\lfloor \frac{3r-1}{2} \right\rfloor$, with equality for $\mathbf{r} + \mathbf{r}$.

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Prop. (Rautenbach [2005]) $\mathrm{ld}(P) \le 2r - 1$.

Pf. Every linear extension L is $(2r-1)$-tight. If $x \| y$, then between them are at most $r-1$ elements incomp. to x $+$ at most $r-1$ incomp. to y, so $|L(x) - L(y)| \le 2r-1$. ∎

Conj. (Tanenbaum-Trenk-Fishburn [2001])
Always $\mathrm{ld}(P) \le \left\lfloor \frac{3r-1}{2} \right\rfloor$, with equality for $\mathbf{r} + \mathbf{r}$.

True for $r = 2$ (Rautenbach [2005]).
True for disconnected posets (Keller-Young [2010]).
True for interval orders ($\mathrm{ld}(P) \le r$) (Keller-Young [2010]).
True for posets of width 2 (this talk).

Upper Bounds

Def. The incomparability graph $G(P)$ has the elements of P as vertices, with $xy \in E(G)$ if $x \| y$ in P. Henceforth let r denote $\Delta(G(P))$.

Prop. (Rautenbach [2005]) $\mathrm{ld}(P) \le 2r-1$.

Pf. Every linear extension L is $(2r-1)$-tight. If $x \| y$, then between them are at most $r-1$ elements incomp. to x + at most $r-1$ incomp. to y, so $|L(x)-L(y)| \le 2r-1$. ■

Conj. (Tanenbaum-Trenk-Fishburn [2001])
Always $\mathrm{ld}(P) \le \left\lfloor \frac{3r-1}{2} \right\rfloor$, with equality for $\mathbf{r}+\mathbf{r}$.

True for $r=2$ (Rautenbach [2005]).
True for disconnected posets (Keller-Young [2010]).
True for interval orders ($\mathrm{ld}(P) \le r$) (Keller-Young [2010]).
True for posets of width 2 (this talk).

However, the conjecture in general is false (this talk).

Posets with Large Linear Discrepancy

Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Posets with Large Linear Discrepancy

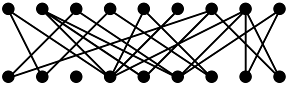
Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Idea: Probabilistic construction produces a poset P consisting of n maximal elts and n minimal elts, with $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$.

Posets with Large Linear Discrepancy

Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Idea: Probabilistic construction produces a poset P consisting of n maximal elts and n minimal elts, with $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$.

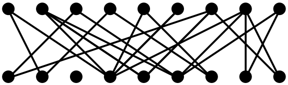


Pf. Minimal elts $x_1, \ldots, x_n$, maximal elts $y_1, \ldots, y_n$.
Let $\mathbb{P}[x_i < y_j] = 1 - p$.

Posets with Large Linear Discrepancy

Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Idea: Probabilistic construction produces a poset P consisting of n maximal elts and n minimal elts, with $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$.



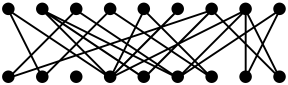
Pf. Minimal elts $x_1, \ldots, x_n$, maximal elts $y_1, \ldots, y_n$.
Let $\mathbb{P}[x_i < y_j] = 1 - p$.

For any element v, $\mathbb{E}[d_{G(P)}(v)] = (n-1) + pn$.

Posets with Large Linear Discrepancy

Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Idea: Probabilistic construction produces a poset P consisting of n maximal elts and n minimal elts, with $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$.



Pf. Minimal elts $x_1, \ldots, x_n$, maximal elts $y_1, \ldots, y_n$.
Let $\mathbb{P}[x_i < y_j] = 1 - p$.

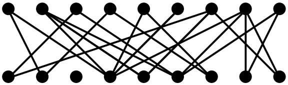
For any element v, $\mathbb{E}[d_{G(P)}(v)] = (n-1) + pn$.

Also $\mathbb{P}[d_{G(P)}(v) > (n-1) + 2pn] \to 0$ exponentially fast.

Posets with Large Linear Discrepancy

Thm. Posets P exist with $\mathrm{ld}(P) \geq 2r - o(r)$.

Idea: Probabilistic construction produces a poset P consisting of n maximal elts and n minimal elts, with $r \leq n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \geq 2n - 2\sqrt{n \ln n}$.



Pf. Minimal elts $x_1, \ldots, x_n$, maximal elts $y_1, \ldots, y_n$.
Let $\mathbb{P}[x_i < y_j] = 1 - p$.

For any element v, $\mathbb{E}[d_{G(P)}(v)] = (n-1) + pn$.

Also $\mathbb{P}[d_{G(P)}(v) > (n-1) + 2pn] \to 0$ exponentially fast.

Multiplied by $2n$ still $\to 0$, so $\mathbb{P}[r > n - 1 + 2pn] \to 0$.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1 - p)^{m^2}$.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1-p)^{m^2}$.

$\mathbb{P}[$some S completely under some $T] \leq \binom{n}{m}^2 (1-p)^{m^2}$

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1-p)^{m^2}$.

$\mathbb{P}[$some S completely under some $T] \le \binom{n}{m}^2 (1-p)^{m^2}$

With $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$ and $1 - p \le e^{-p}$, the probability is bounded by $\left(\frac{ne}{m}\right)^{2m} e^{-pm^2}$.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1-p)^{m^2}$.

$\mathbb{P}[$some S completely under some $T] \le \binom{n}{m}^2 (1-p)^{m^2}$

With $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$ and $1 - p \le e^{-p}$, the probability is bounded by $\left(\frac{ne}{m}\right)^{2m} e^{-pm^2}$.

This tends to 0 when $m = \sqrt{n \ln n}$ and $p = 3\sqrt{(\ln n)/n}$.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1-p)^{m^2}$.

$\mathbb{P}[$some S completely under some $T] \le \binom{n}{m}^2 (1-p)^{m^2}$

With $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$ and $1 - p \le e^{-p}$, the probability is bounded by $\left(\frac{ne}{m}\right)^{2m} e^{-pm^2}$.

This tends to 0 when $m = \sqrt{n \ln n}$ and $p = 3\sqrt{(\ln n)/n}$.

So, with high probability, every linear extension of the resulting P has some element among the first $\sqrt{n \ln n}$ incomparable to some element among the last $\sqrt{n \ln n}$.

Lower Bound on $\mathrm{ld}(P)$

Let S and T be m-sets of minimal and maximal elts.

$\mathbb{P}[S < T] = (1-p)^{m^2}$.

$\mathbb{P}[$some S completely under some $T] \le \binom{n}{m}^2 (1-p)^{m^2}$

With $\binom{n}{m} \le \left(\frac{ne}{m}\right)^m$ and $1 - p \le e^{-p}$, the probability is bounded by $\left(\frac{ne}{m}\right)^{2m} e^{-pm^2}$.

This tends to 0 when $m = \sqrt{n \ln n}$ and $p = 3\sqrt{(\ln n)/n}$.

So, with high probability, every linear extension of the resulting P has some element among the first $\sqrt{n \ln n}$ incomparable to some element among the last $\sqrt{n \ln n}$.

W.h.p., $r \le n + 6\sqrt{n \ln n}$ and $\mathrm{ld}(P) \ge 2n - 2\sqrt{n \ln n}$,
so in this model almost always $\mathrm{ld}(P) \ge 2r - O(\sqrt{r \ln r})$. ∎

Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \leq \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \leq \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Pf. Let $I(x) = \{y \in P\colon\ y \| x\}$.

Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \le \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Pf. Let $I(x) = \{y \in P\colon\ y \| x\}$.

We may assume $I(z) \neq \varnothing$ for all z
[otherwise, $r(P) = r(P-z)$ and $\mathrm{ld}(P) = \mathrm{ld}(P-z)$].

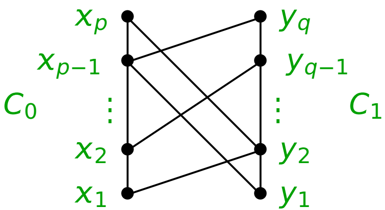
Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \leq \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Pf. Let $I(x) = \{y \in P\colon\ y \| x\}$.

We may assume $I(z) \neq \emptyset$ for all z
[otherwise, $r(P) = r(P-z)$ and $\mathrm{ld}(P) = \mathrm{ld}(P-z)$].

Dilworth's Theorem $\Rightarrow$ P is covered by chains C_0 & C_1.



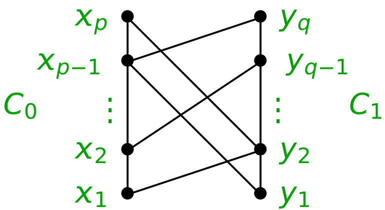
Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \leq \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Pf. Let $I(x) = \{y \in P\colon\ y \| x\}$.

We may assume $I(z) \neq \varnothing$ for all z
[otherwise, $r(P) = r(P-z)$ and $\mathrm{ld}(P) = \mathrm{ld}(P-z)$].

Dilworth's Theorem $\Rightarrow$ P is covered by chains C_0 & C_1.



$I(z)$ is an interval on the chain C_k not containing z.

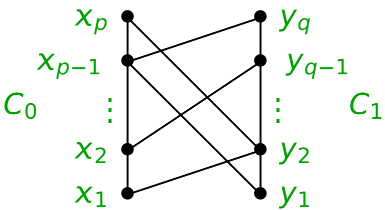
Posets of Width 2

Thm. If P has width 2, then $\mathrm{ld}(P) \leq \left\lfloor \frac{3r-1}{2} \right\rfloor$.

Pf. Let $I(x) = \{y \in P\colon\ y \| x\}$.

We may assume $I(z) \neq \emptyset$ for all z
[otherwise, $r(P) = r(P-z)$ and $\mathrm{ld}(P) = \mathrm{ld}(P-z)$].

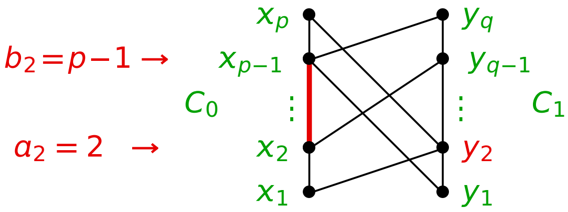
Dilworth's Theorem $\Rightarrow$ P is covered by chains C_0 & C_1.



$I(z)$ is an interval on the chain C_k not containing z.

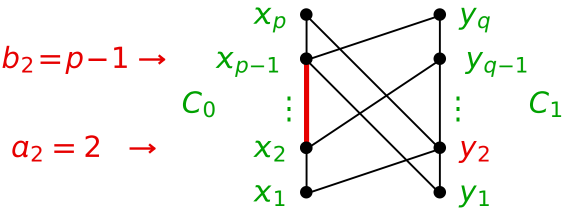
Use these intervals to define the linear extension.

Construction of the Linear Extension



Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$.

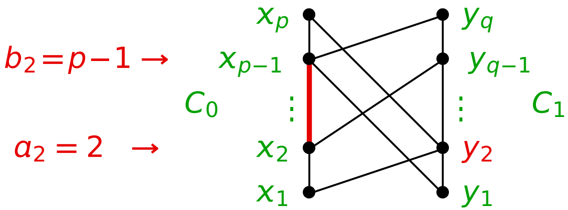
Construction of the Linear Extension



Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$.

- $a_1 \leq \cdots \leq a_q$ and $b_1 \leq \cdots \leq b_q$

Construction of the Linear Extension

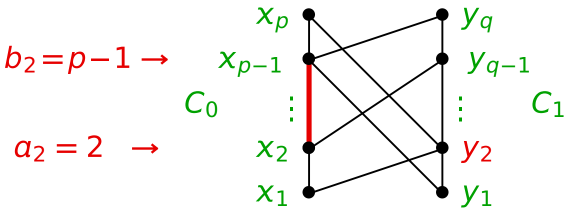


Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$.

- $a_1 \leq \cdots \leq a_q$ and $b_1 \leq \cdots \leq b_q$

$\therefore$ $\frac{a_{j+1}+b_{j+1}}{2} \geq \frac{a_j+b_j}{2}$ for $1 \leq j < q$.

Construction of the Linear Extension



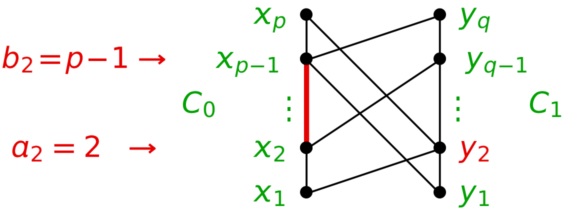
Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$.

- $a_1 \leq \cdots \leq a_q$ and $b_1 \leq \cdots \leq b_q$

$\therefore$ $\frac{a_{j+1}+b_{j+1}}{2} \geq \frac{a_j+b_j}{2}$ for $1 \leq j < q$.

Form a linear extension L of P by inserting y_j between x_{s_j} and x_{1+s_j} on C_0, where $s_j = \left\lfloor \frac{a_j+b_j}{2} \right\rfloor$.

Construction of the Linear Extension



Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$.

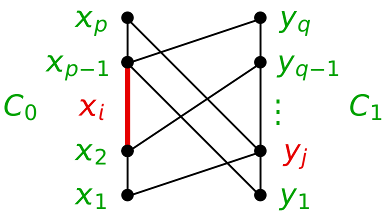
- $a_1 \leq \cdots \leq a_q$ and $b_1 \leq \cdots \leq b_q$

$\therefore$ $\frac{a_{j+1}+b_{j+1}}{2} \geq \frac{a_j+b_j}{2}$ for $1 \leq j < q$.

Form a linear extension L of P by inserting y_j between x_{s_j} and x_{1+s_j} on C_0, where $s_j = \left\lfloor \frac{a_j+b_j}{2} \right\rfloor$.

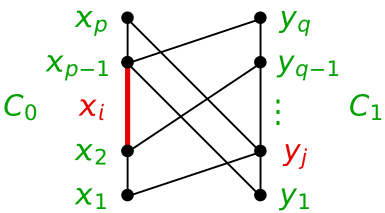
It remains to show that if $x_i \| y_j$, then at most $\frac{3r-1}{2} - 1$ elements lie between them on L.

Analysis of Tightness



Fix $x_i \| y_j$. Let m_k be the number of elements of C_k between x_i and y_j on L; we want $m_0 + m_1 \leq \left\lfloor \frac{3r-1}{2} \right\rfloor - 1$.

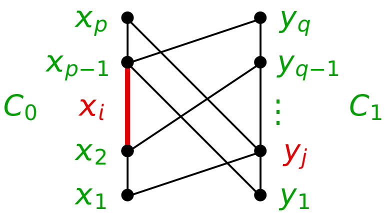
Analysis of Tightness



Fix $x_i \| y_j$. Let m_k be the number of elements of C_k between x_i and y_j on L; we want $m_0 + m_1 \leq \left\lfloor \frac{3r-1}{2} \right\rfloor - 1$.

Since $x_i \| y_j$, every element of C_1 between y_j and x_i is incomparable to x_i, as is y_j; hence $m_1 \leq r - 1$.

Analysis of Tightness

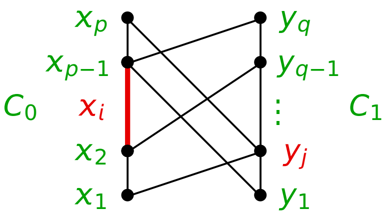


Fix $x_i \| y_j$. Let m_k be the number of elements of C_k between x_i and y_j on L; we want $m_0 + m_1 \leq \left\lfloor \frac{3r-1}{2} \right\rfloor - 1$.

Since $x_i \| y_j$, every element of C_1 between y_j and x_i is incomparable to x_i, as is y_j; hence $m_1 \leq r - 1$.

Since $x_i \in I(y_j)$, the placement of y_j just above x_{s_j} within C_0 guarantees $m_0 \leq \left\lfloor \frac{b_j - a_j}{2} \right\rfloor$.

Analysis of Tightness



Fix $x_i \| y_j$. Let m_k be the number of elements of C_k between x_i and y_j on L; we want $m_0 + m_1 \leq \left\lfloor \frac{3r-1}{2} \right\rfloor - 1$.

Since $x_i \| y_j$, every element of C_1 between y_j and x_i is incomparable to x_i, as is y_j; hence $m_1 \leq r - 1$.

Since $x_i \in I(y_j)$, the placement of y_j just above x_{s_j} within C_0 guarantees $m_0 \leq \left\lfloor \frac{b_j - a_j}{2} \right\rfloor$. Since $b_j - a_j \leq r - 1$,

$$m_0 + m_1 \leq \left\lfloor \frac{b_j - a_j}{2} \right\rfloor + r - 1 \leq \left\lfloor \frac{3(r-1)}{2} \right\rfloor. \qquad \blacksquare$$

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Every linear extension gives a vertex ordering for $G(P)$. Thus $\mathrm{ld}(P) \geq B(G(P))$, but could $B(G(P))$ be smaller using an ordering not arising from a linear extension?

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Every linear extension gives a vertex ordering for $G(P)$. Thus $\mathrm{ld}(P) \geq B(G(P))$, but could $B(G(P))$ be smaller using an ordering not arising from a linear extension?

Thm. (FTT [2001]) $\mathrm{ld}(P) = B(G(P))$.

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Every linear extension gives a vertex ordering for $G(P)$. Thus $\mathrm{ld}(P) \geq B(G(P))$, but could $B(G(P))$ be smaller using an ordering not arising from a linear extension?

Thm. (FTT [2001]) $\mathrm{ld}(P) = B(G(P))$.
(Brightwell [unpub.] gave another proof.)

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Every linear extension gives a vertex ordering for $G(P)$. Thus $\mathrm{ld}(P) \geq B(G(P))$, but could $B(G(P))$ be smaller using an ordering not arising from a linear extension?

Thm. (FTT [2001]) $\mathrm{ld}(P) = B(G(P))$.
(Brightwell [unpub.] gave another proof.)

But, approximation is easy. Take any linear extension!

Complexity

Thm. (TTF [2001]) Computing $\mathrm{ld}(P)$ is NP-complete.

Def. The bandwidth $B(G)$ of a graph G is the least t such that $V(G)$ has a numbering $v_1, \ldots, v_n$ in which labels of adjacent vertices differ by at most t.

Every linear extension gives a vertex ordering for $G(P)$. Thus $\mathrm{ld}(P) \geq B(G(P))$, but could $B(G(P))$ be smaller using an ordering not arising from a linear extension?

Thm. (FTT [2001]) $\mathrm{ld}(P) = B(G(P))$.
(Brightwell [unpub.] gave another proof.)

But, approximation is easy. Take any linear extension!

Thm. If L is any linear extension of P, then $t(L) \leq 3\,\mathrm{ld}(P)$, with inequality infinitely often.

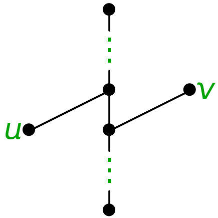
Sharpness example (Rautenbach [2005])

Construct P with linear extensions L and L' such that $t(L) = 3t(L')$.

Sharpness example (Rautenbach [2005])

Construct P with linear extensions L and L' such that $t(L) = 3t(L')$.

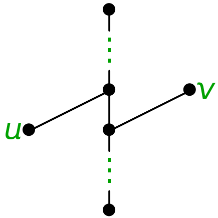
Take $|P| = 2k + 2$ with $k \equiv 1 \mod 3$; put u below the top half and v above the bottom half of a $2k$-chain.



Sharpness example (Rautenbach [2005])

Construct P with linear extensions L and L' such that $t(L) = 3t(L')$.

Take $|P| = 2k + 2$ with $k \equiv 1 \mod 3$; put u below the top half and v above the bottom half of a $2k$-chain.



L	$u <$ chain $< v$	$(2k+1)$-tight
L'	$\frac{2k+1}{3} < u < \frac{2k-2}{3} < v < \frac{2k+1}{3}$	$\frac{2k+1}{3}$-tight

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

Let $t' = \mathrm{ld}(P')$, with L' a t'-tight linear extension of P'.
Note that $P' \subseteq P$ implies $\mathrm{ld}(P') \leq \mathrm{ld}(P)$, so $t' \leq \mathrm{ld}(P)$.

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

Let $t' = \mathrm{ld}(P')$, with L' a t'-tight linear extension of P'.
Note that $P' \subseteq P$ implies $\mathrm{ld}(P') \leq \mathrm{ld}(P)$, so $t' \leq \mathrm{ld}(P)$.

Elements below x on L' are in $I(x)$; above y are in $I(y)$.

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

Let $t' = \mathrm{ld}(P')$, with L' a t'-tight linear extension of P'.
Note that $P' \subseteq P$ implies $\mathrm{ld}(P') \leq \mathrm{ld}(P)$, so $t' \leq \mathrm{ld}(P)$.

Elements below x on L' are in $I(x)$; above y are in $I(y)$.

Either $S_0 < y < S_1 < x < S_2$ on L'
or $S_0 < x < S_1 < y < S_2$ on L'.

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

Let $t' = \mathrm{ld}(P')$, with L' a t'-tight linear extension of P'.
Note that $P' \subseteq P$ implies $\mathrm{ld}(P') \leq \mathrm{ld}(P)$, so $t' \leq \mathrm{ld}(P)$.

Elements below x on L' are in $I(x)$; above y are in $I(y)$.

Either $S_0 < y < S_1 < x < S_2$ on L'
or $S_0 < x < S_1 < y < S_2$ on L'.

In either case, $t' \geq |S_0|$, $t' \geq |S_2|$, and $t' \geq |S_1| + 1$.

The Bound

Fix an extension L of P, with $x \| y$ and $t = L(y) - L(x)$.

$$L: \qquad A < \underbrace{x < B < y}_{P'} < C$$

Let $t' = \mathrm{ld}(P')$, with L' a t'-tight linear extension of P'.
Note that $P' \subseteq P$ implies $\mathrm{ld}(P') \leq \mathrm{ld}(P)$, so $t' \leq \mathrm{ld}(P)$.

Elements below x on L' are in $I(x)$; above y are in $I(y)$.

Either $S_0 < y < S_1 < x < S_2$ on L'
or $S_0 < x < S_1 < y < S_2$ on L'.

In either case, $t' \geq |S_0|$, $t' \geq |S_2|$, and $t' \geq |S_1| + 1$.

$\therefore\ t = |P'| - 1 = |S_0| + |S_1| + |S_2| + 1 \leq 3t' \leq 3\mathrm{ld}(P)$. ■

Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
$\mathrm{ld}(\mathbf{2}^n) = 2^n - 2^{(n+1)/2} - 1$ for odd n.

Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
$\mathrm{ld}(\mathbf{2}^n) = 2^n - 2^{(n+1)/2} - 1$ for odd n.

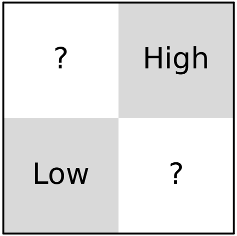
Thm. (Hong-Hyun-Kim-Kim [2005]) $\mathrm{ld}(\mathbf{m} \times \mathbf{n}) = \left\lceil \frac{mn}{2} \right\rceil - 2$.

Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
$\mathrm{ld}(\mathbf{2}^n) = 2^n - 2^{(n+1)/2} - 1$ for odd n.

Thm. (Hong-Hyun-Kim-Kim [2005]) $\mathrm{ld}(\mathbf{m} \times \mathbf{n}) = \left\lceil \frac{mn}{2} \right\rceil - 2$.

General idea:

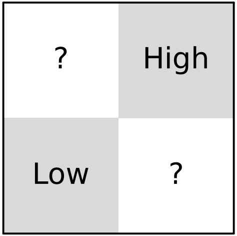


Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
$\mathrm{ld}(\mathbf{2}^n) = 2^n - 2^{(n+1)/2} - 1$ for odd n.

Thm. (Hong-Hyun-Kim-Kim [2005]) $\mathrm{ld}(\mathbf{m} \times \mathbf{n}) = \left\lceil \frac{mn}{2} \right\rceil - 2$.

General idea:



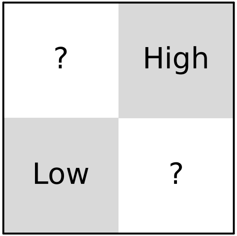
Thm. (Choi [2008], Kim–Cheong [2008])
$\mathrm{ld}(\mathbf{k}^3) = \frac{3}{4}k^3 - \frac{1}{2}k^2 - 1$ when k is even.

Products of Chains

Thm. (TTF [2001]) $\mathrm{ld}(\mathbf{2}^n) = 2^n - 3 \cdot 2^{n/2}$ for even n.
$\mathrm{ld}(\mathbf{2}^n) = 2^n - 2^{(n+1)/2} - 1$ for odd n.

Thm. (Hong-Hyun-Kim-Kim [2005]) $\mathrm{ld}(\mathbf{m} \times \mathbf{n}) = \left\lceil \frac{mn}{2} \right\rceil - 2$.

General idea:



Thm. (Choi [2008], Kim–Cheong [2008])
$\mathrm{ld}(\mathbf{k}^3) = \frac{3}{4}k^3 - \frac{1}{2}k^2 - 1$ when k is even.

Thm. (Choi-West) The general upper bound $\mathrm{ld}(\mathbf{k}^d) \le (1 - 2^{-d+1})k^d + O(k^{d-1})$ is sharp up to the lower-order term when $d = 4$.