The Dimension of Posets with Planar Cover Graphs

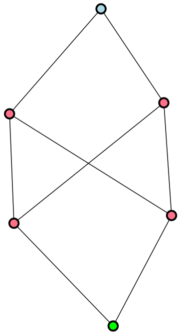
Veit Wiechert

Co-Authors

William T. Trotter

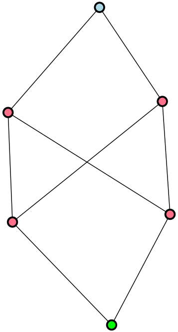
Stefan Felsner

Drawing Posets



Diagram

Drawing Posets



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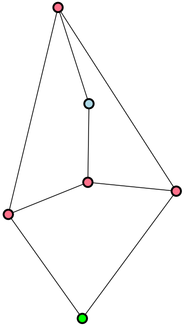
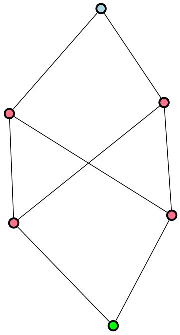


Cover Graph

Drawing Posets



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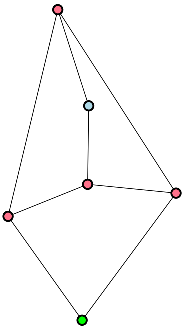

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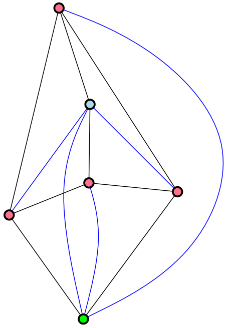

Comparability Graph

Dimension and the Standard Example S_n

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$$\mathbf{P} = \bigcap_{L \in R} L$$

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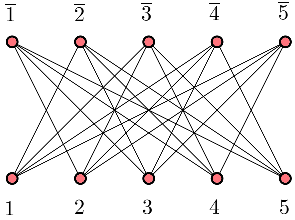
S_5



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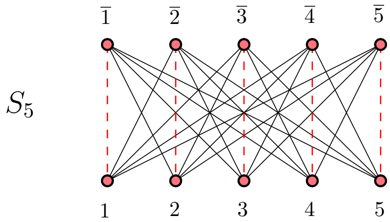

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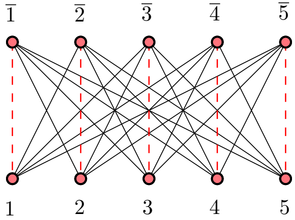
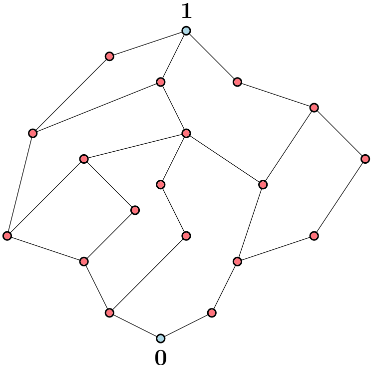


$$\boxed{\dim(S_n) = n}$$

Theorem (Baker, Fishburn, Roberts '71)

If **P** *is planar and has a zero and a one, then*

$$\dim(\mathbf{P}) \leq 2$$



Theorem (Trotter, Moore '77)

If **P** *is planar and has a zero, then*

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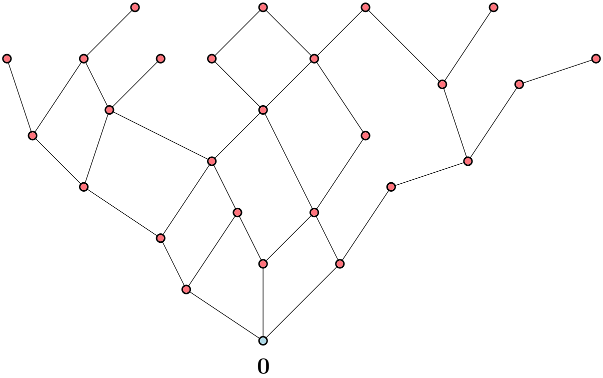
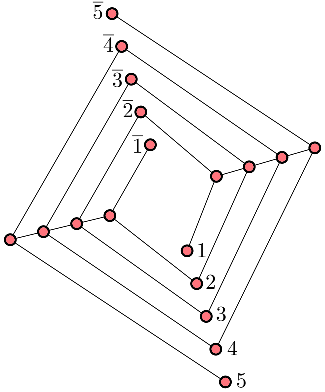


Kelly's example '81



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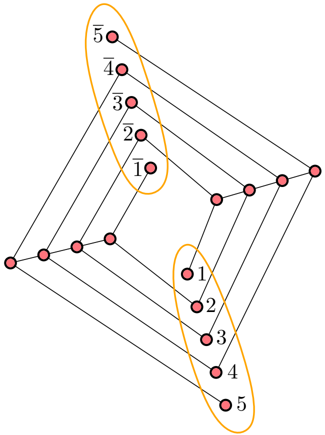

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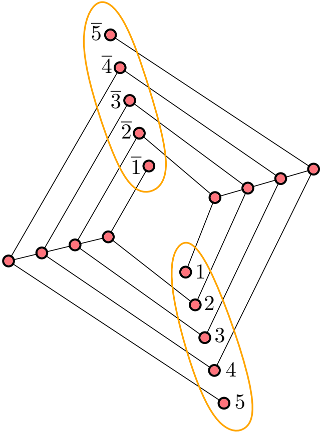

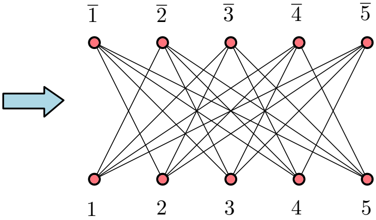

Which Conditions bound the Dimension?

Theorem (Felsner, Li, Trotter 2010)

If $\mathbf{P}$ *has a planar cover graph and* $\mathrm{h}(\mathbf{P}) \leq 2$ *then*

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Theorem (Streib, Trotter 2011)

For every $\mathbf{h}$ *there exist a constant* $c_{\mathbf{h}}$*, s.t. if* $\mathbf{P}$ *has height* $\mathbf{h}$ *and a planar cover graph, then*

$$\dim(\mathbf{P}) \leq c_{\mathbf{h}}$$

Boxicity and Poset Dimension

$\mathrm{box}(G) := \min k$ such that $\exists$ k-box representation of G.

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Corollary

If $G_{\mathbf{P}}$ *is planar, then*

$$\dim(\mathbf{P}) \leq 6$$

Our Results

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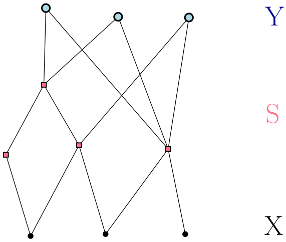
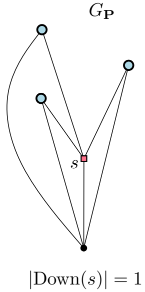
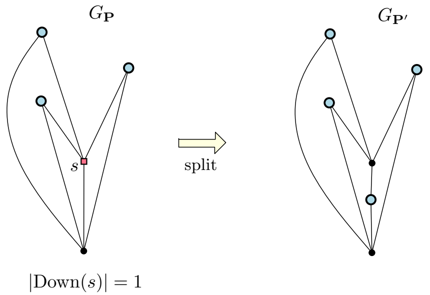
- case $\mathrm{h}(\mathbf{P}) \leq 2$: Apply (Felsner, Li, Trotter)
- case $\mathrm{h}(\mathbf{P}) = 3$ or 4:




$G_{\mathbf{P}}$
s
$|\mathrm{Down}(s)| = 1$


$G_{\mathbf{P}}$
s
split
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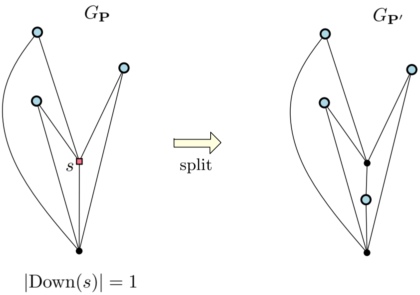
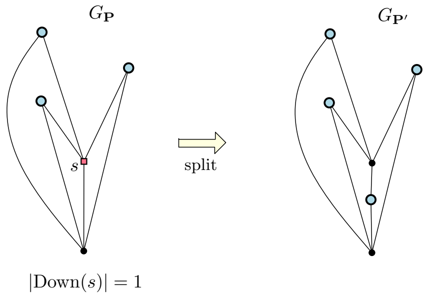
$G_{\mathbf{P}'}$



- $G_{\mathbf{P'}}$ is planar



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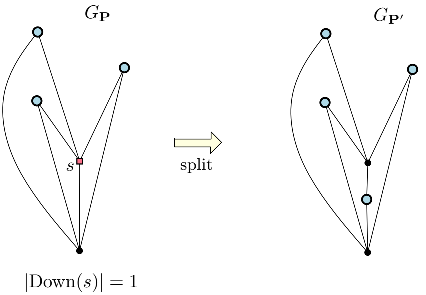

- $G_{\mathbf{P}'}$ is planar
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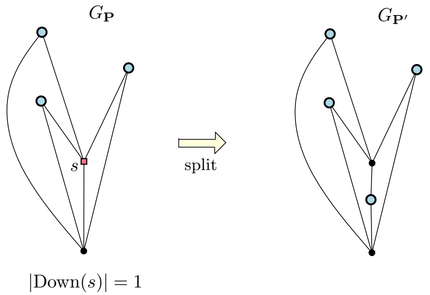
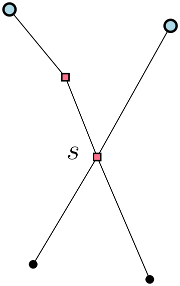
- similar for the case: $|\mathrm{Up}(s)| = 1$



- $G_{\mathbf{P'}}$ is planar
- $\dim(\mathbf{P}) \leq \dim(\mathbf{P'})$
- similar for the case: $|\mathrm{Up}(s)| = 1$
- this leads to the case:

$$|\mathrm{Down}(s)| \geq 2 \text{ and } |\mathrm{Up}(s)| \geq 2 \text{ for every } s \in S$$

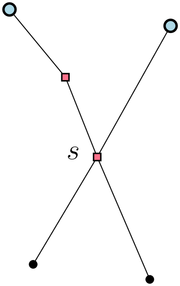
Diagram



$$|\mathrm{Down}(s)| = 2$$

$$|\mathrm{Up}(s)| = 3$$

Diagram



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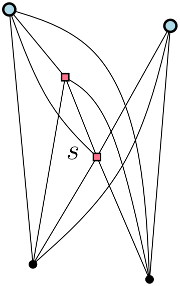
Comparability Graph



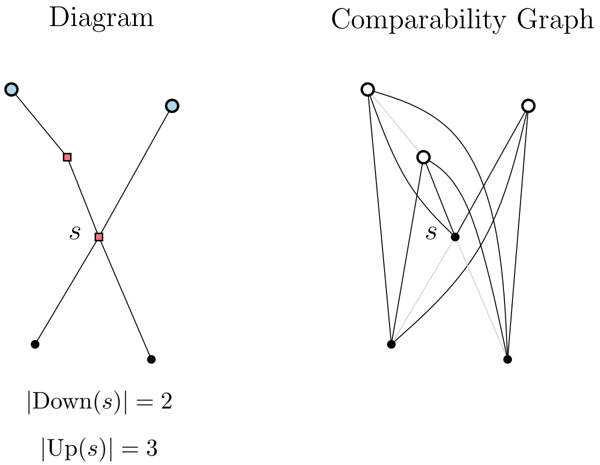


Diagram
Comparability Graph
s
s
|Down(s)| = 2
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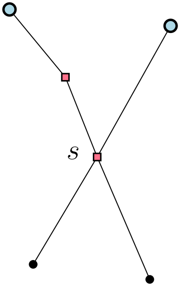
Diagram



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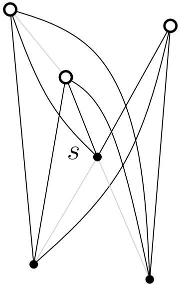
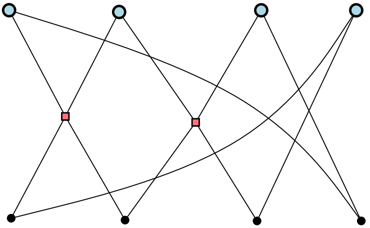
Comparability Graph



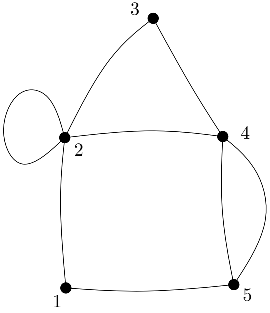
$\Rightarrow G_{\mathbf{P}}$ is not planar

Final Case



$$|\mathrm{Down}(s)| = |\mathrm{Up}(s)| = 2 \quad \text{for every } s \in S$$

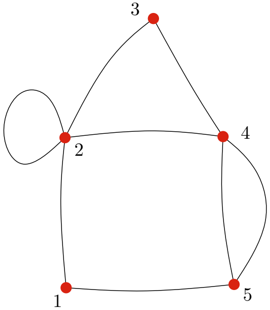
Brightwell-Trotter-Theorem '96



G

P_G

Brightwell-Trotter-Theorem '96



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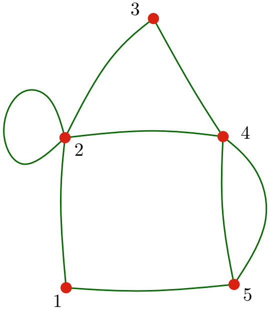

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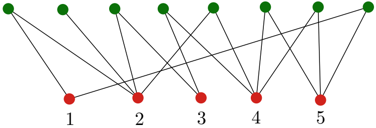

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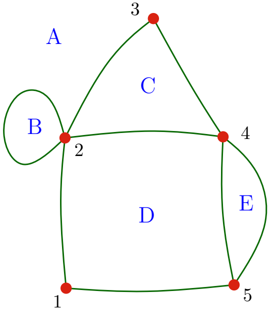

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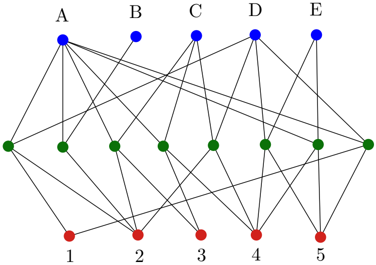

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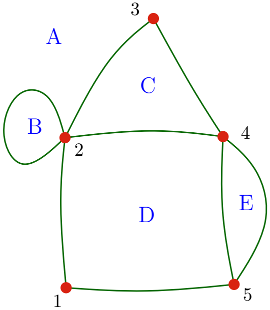

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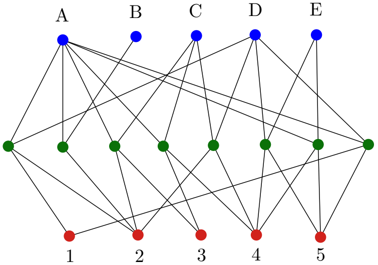
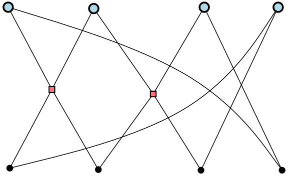


P_G

$$\dim(P_G) \leq 4$$

Posets with Planar Comparability Graphs



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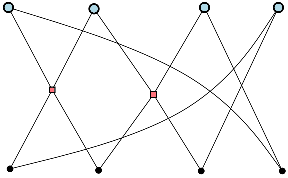

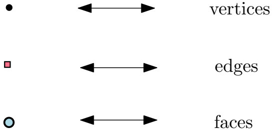

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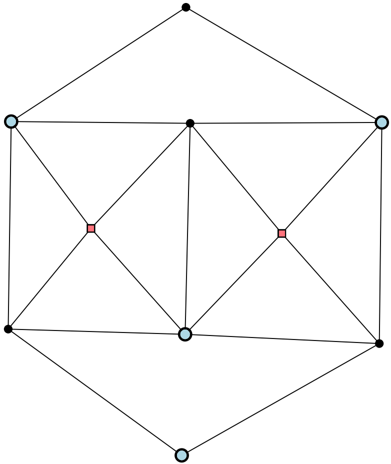

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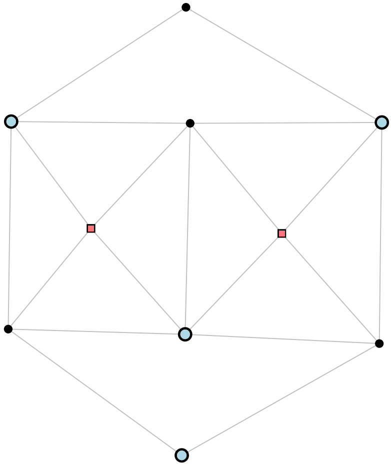

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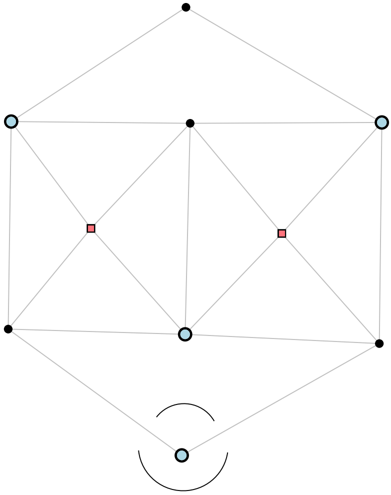

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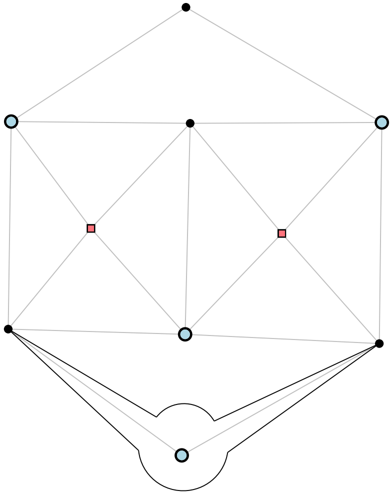

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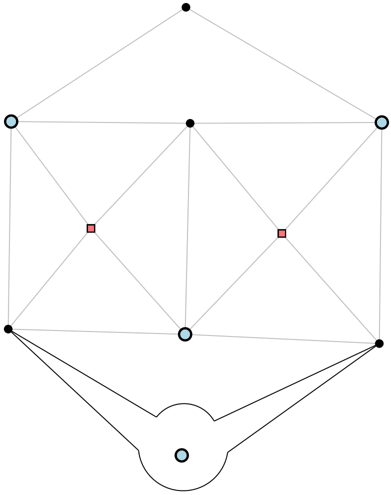

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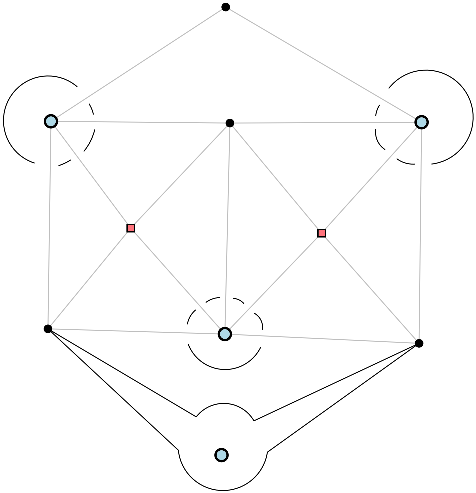

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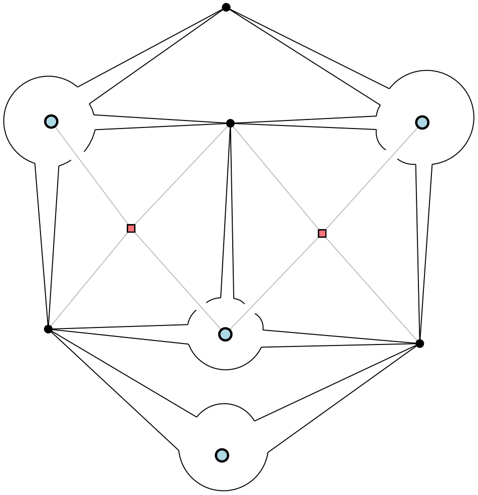

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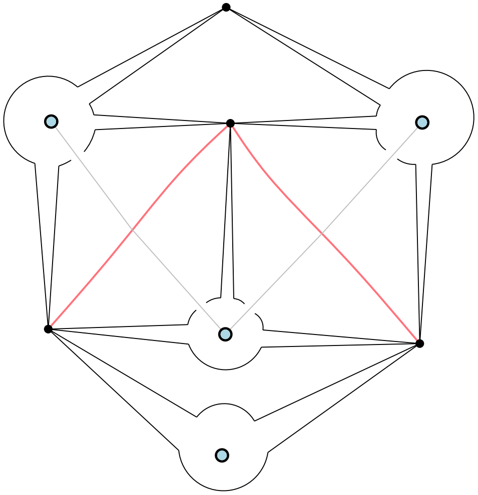

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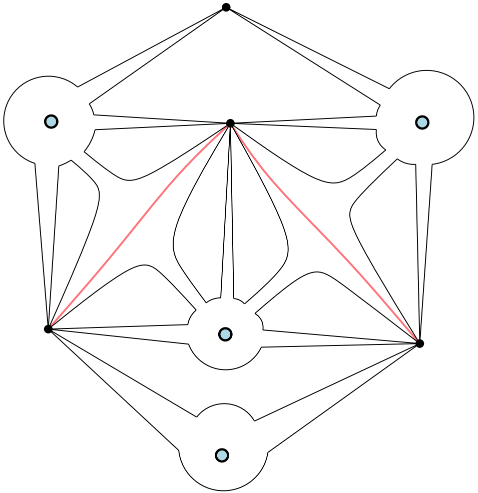

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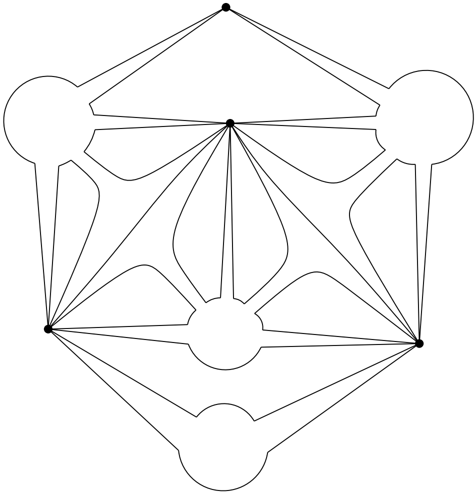

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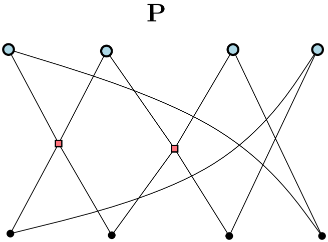

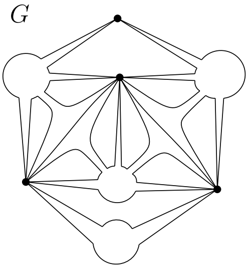

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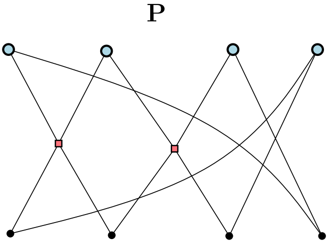

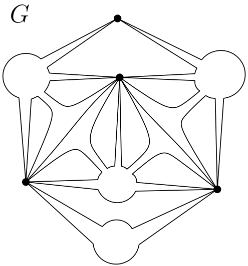

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$$\Rightarrow \dim(\mathbf{P}) = \dim(\mathbf{P}') \leq \dim(\mathbf{P}_G) \underset{\text{(BTT)}}{\leq} 4$$

Posets with Outerplanar Cover Graphs

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$$\dim(\mathbf{P}) \leq 4$$

Theorem

If $\mathbf{P}$ *has an outerplanar cover graph and* $\mathrm{h}(\mathbf{P}) \leq 2$*, then*

$$\dim(\mathbf{P}) \leq 3$$

Posets with Outerplanar Cover Graphs

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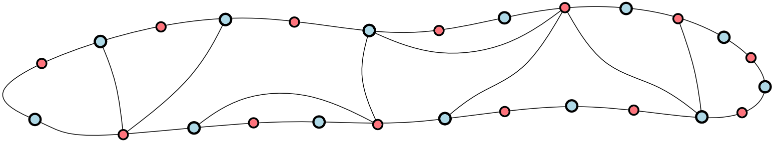
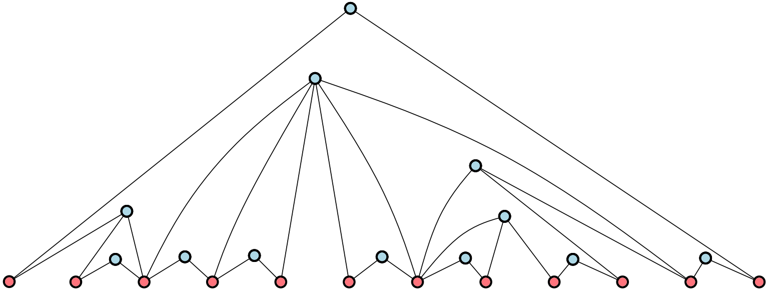
Fact: Both inequalities are best possible

Outerplanar Cover Graph and $\mathrm{h}(\mathbf{P}) \leq 2$



Outerplanar Cover Graph and $\mathrm{h}(\mathbf{P}) \leq 2$



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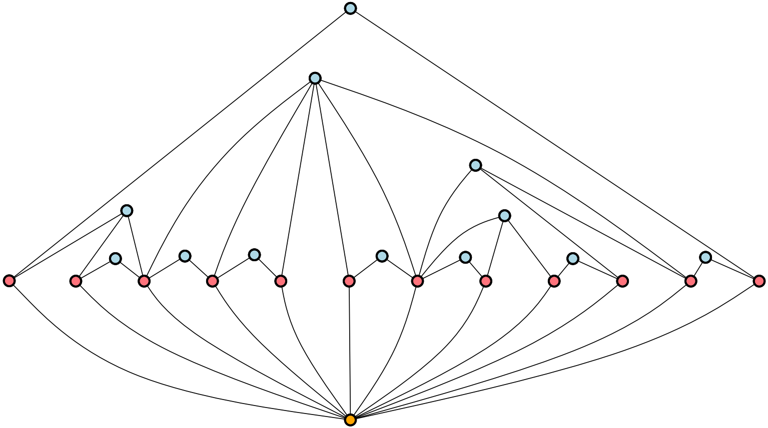
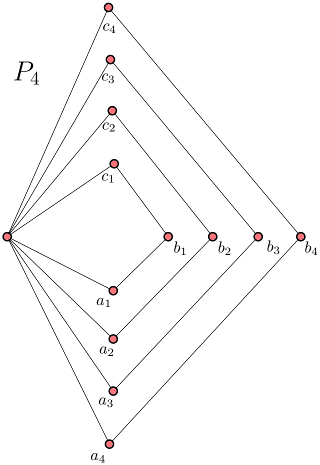


Lower Bound

P_4



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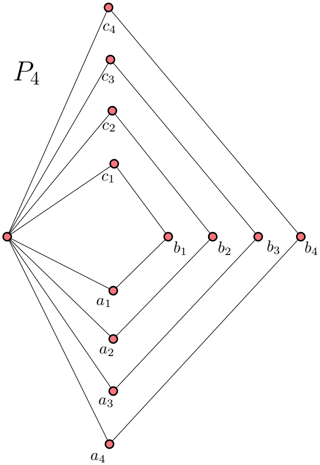

- If $n \geq 17$ then $\dim(P_n) = 4$

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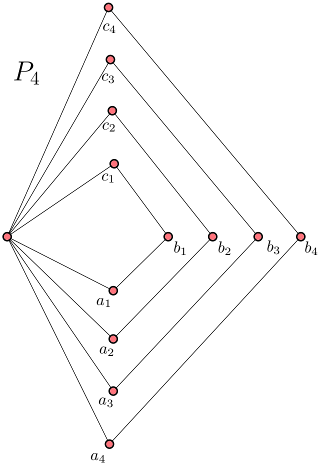

- If $n \geq 17$ then $\dim(P_n) = 4$
- applied (Erdős, Szekeres)
  $17 = 4^2 + 1$ and $5 = 2^2 + 1$

Open Problems

- better bounds for $\dim(\mathbf{P})$ when $\mathbf{P}$ is planar and $\mathrm{h}(\mathbf{P}) \leq k$

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Open Problems

- better bounds for $\dim(\mathbf{P})$ when $\mathbf{P}$ is planar and $\mathrm{h}(\mathbf{P}) \leq k$
- is it *NP*-complete to decide whether **P** is a subposet of a poset with a planar cover graph?
- Are there t_n s.t. if **P** is planar with $\dim(\mathbf{P}) \geq t_n$ then **P** contains S_n as a subposet?

Thank you for your attention

Bonusmaterial: nonplanar Poset with outerplanar Cover Graph