First-Fit chromatic number of interval graphs

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Outline

Overview

Lower bound: A direct construction

Upper bound: Column construction method

$FF(k)$

A first-fit coloring of a graph G is to color each vertex of G a positive integer in a way such that each vertex with color i has a neighbor assigned color j for every $j = 1, \ldots, i - 1$ and has no neighbor of color i. (This color set partition is called a "wall" in the graph.)

The first-fit chromatic number, also called the Grundy number, of a graph, is the maximum possible number used in a first-fit coloring of the graph. This parameter is just the number of colors needed in the worst case when applying the greedy online coloring algorithm First-Fit on a graph.

When talking about a first-fit coloring of a family of intervals we indeed refer to the first-fit coloring of its intersection graph. Let $FF(k)$ denote the largest first-fit chromatic number of an interval graph whose maximum clique size is k.

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Lower bound

In 1990, Chrobak and Slusarek proved that $\boxed{FF(k) \geq 4k-9}$*, when* $k \geq 4$*, ...*
`http://people.math.gatech.edu/~trotter/rprob.html`

Theorem 1

$FF(k) \geq 4k - 5$ for any positive integer k.

Upper bound

In 2003, Pemmaraju, Raman and Varadarajan made a major breakthrough by showing that $FF(k) \leq 10k$ *and commented that their upper bound might be improved but that the technique wouldn't yield a result better than 8k. Later in 2003, their predictions were confirmed, and their technique was refined by Brightwell, Kierstead and Trotter to obtain an upper bound of 8k. In 2004, Narayansamy and Babu found an even cleaner argument for this bound that actually yields the slightly stronger result:* $FF(k) \leq 8k - 3$*. Howard has recently pointed out that one can actually show that* $\boxed{FF(k) \leq 8k - 4}$*.*

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Theorem 2

$FF(k) \leq 8k - 9$ *for* $k \geq 2$.

Asymptotic lower bound

Later in 2004, Kierstead and Trotter gave a computer proof that $FF(k) \geq 4.99k - C$. This technique was subsequently refined to show that $FF(k) \geq 4.99999k - C$. And in 2009, D. Smith showed that for every $\boxed{e > 0, FF(k) > (5-e)k}$, *when k is sufficiently large.*

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Asymptotic upper bound

Theorem 3

For every $e > 0, FF(k) < 8k - (2 - e)\log_3 k$*, when* k *is sufficiently large.*

$\lim_{k\to\infty} FF(k)/k = 5$?

So as k goes to infinity, the ratio $FF(k)/k$ tends to a limit that is somewhere between 5 and 8. I will bet a nice bottle of wine that 5 is the right answer.

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We display a direct construction to establish Theorem 1.

We make some simple observations on the Column Construction Method invented by Pemmaraju, Raman, Varadarajan (2003) and these simple observations will lead to Theorems 2 and 3.

Outline

$FF(1) = 1 > -1 = 4 - 5$

$FF(2) = 4 > 3 = 8 - 5$

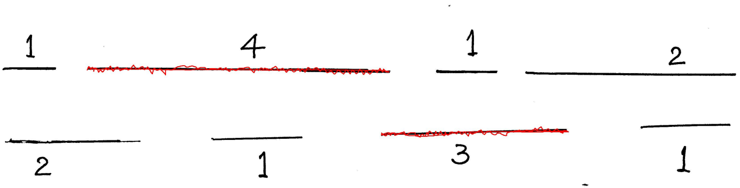

Figure: M_2

$FF(3) = 12 - 5 = 7$

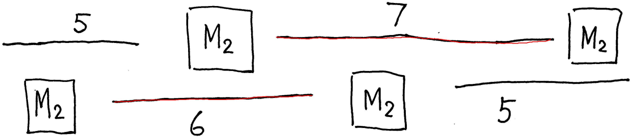

Figure: M_3

$FF(4) \geq 16 - 5 = 11$

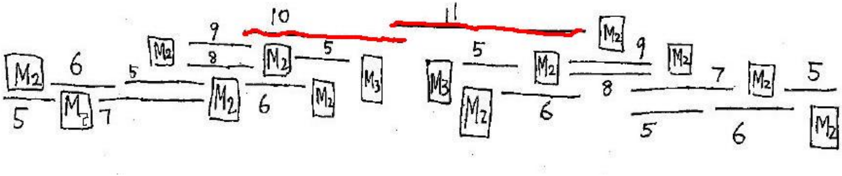

Figure: M_4

$FF(5) \geq 20 - 5 = 15$

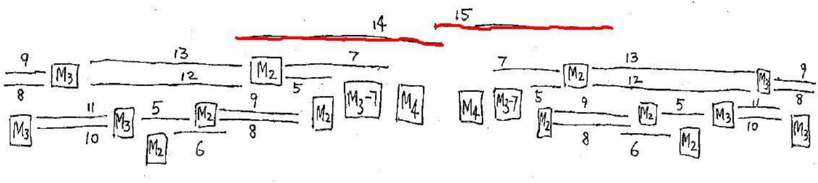

Figure: M_5

$FF(6) \geq 24 - 5 = 19$

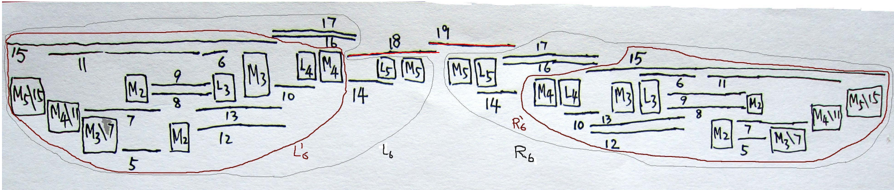

Figure: M_6

$FF(n+1) \geq 4n-1$

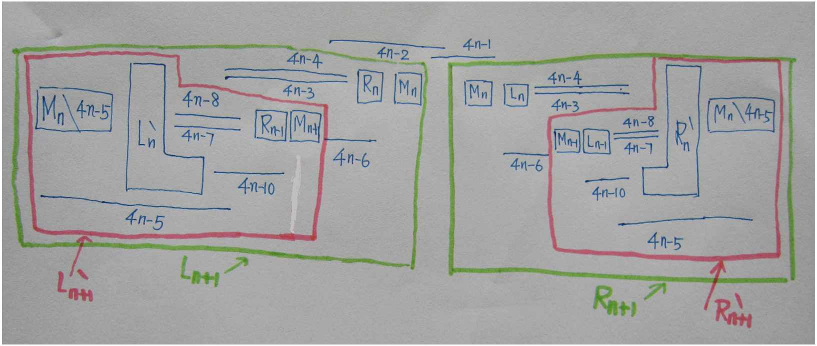

Figure: $M_{n+1}, n \geq 6$

Outline

Overview

Lower bound: A direct construction

Upper bound: Column construction method

We basically follow N.S. Narayanaswamy and R. Subhash Babu,

A note on first-fit coloring of interval graphs, Order 25 (2008) 49–53

to introduce the Column Construction Method invented by Pemmaraju, Raman, Varadarajan (2003).

Column Construction Method: Beginning Step

Let V be a family of intervals and let G be its intersection graph. Assume that the clique number of G is k. Given a first-fit coloring L of V using $FF(k)$ colors, let us construct some columns (buildings) row by row (floor by floor) each box (room) of them is labelled by $\mathscr{A}$, $\mathscr{B}$, or $\mathscr{C}$.

Lay the Foundation Stone: Take a set of maximal cliques $Q_1, \ldots, Q_r$ of G such that $\cup_{i=1}^{r} Q_i = V(G)$ and Q_i lies to the left of Q_j for $i < j$; (This means that $\cap_{T \in Q_i} T, i = 1, \ldots, r$, are a set of pairwise disjoint intervals and $\cap_{T \in Q_i} T$ lies to the left of $\cap_{T \in Q_j} T$ if $i < j$). For each clique Q_i we construct a basement for the column corresponding to the clique, also denoted Q_i. We view the basement at column i and that at column $i+1$ as neighbors of each other at height 0.

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Column Construction Method: Inductive Procedure

For positive integer i, build the ith row after the $(i-1)$th row is finished. Suppose that column j reaches height $i-1$.

- If there is a vertex in Q_j receiving color i, then add one $\mathscr{A}$-box to column j at height i;
- If column j does not grow to height i with an $\mathscr{A}$-box but at least one of its neighbors at height $i-1$ does so, we add a $\mathscr{B}$-box to column j at height i;
- *Suppose column j does not grow to height i with either an $\mathscr{A}$-box or a $\mathscr{B}$-box. Add a $\mathscr{C}$-box to column j at height i if there is an integer $0 < \ell \leq i-1$ such that the number of $\mathscr{A}$-boxes from height ℓ to $i-1$ in column j is greater than $\frac{i-\ell+1}{4}$.*
- After constructing the three kinds of boxes at height i, we define column p and column q to be neighbors at height i if there is no column growing to height i inbetween them (so they can see each other at height i).

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Bounding Height

Following the proof of N.S. Narayanaswamy and R. Subhash Babu, we have

$\{v : L(v) \geq i\}$ is covered by those cliques growing to the ith floor.

This shows that we can get upper bound of $FF(k)$ by bounding the maximum height of these columns. To get tighter bound, we also choose an $\mathscr{A}$-box on the $FF(k)$th floor and study how many $\mathscr{C}$-boxes can be above it.

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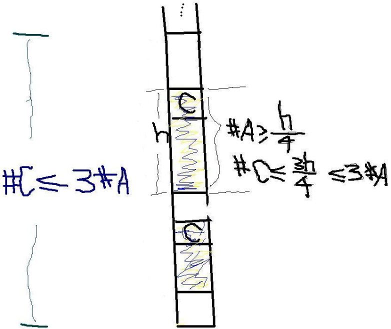
Observation I

Since we cannot add a $\mathscr{C}$-box on the top of a column, among the consecutive highest h boxes from this column the number of $\mathscr{A}$-boxes, denoted $\#A$, satisfy $\#A < \frac{h+1}{4}$ and hence, considering that $\#A$ is an integer, $\boxed{\#A \leq \frac{h}{4}.}$

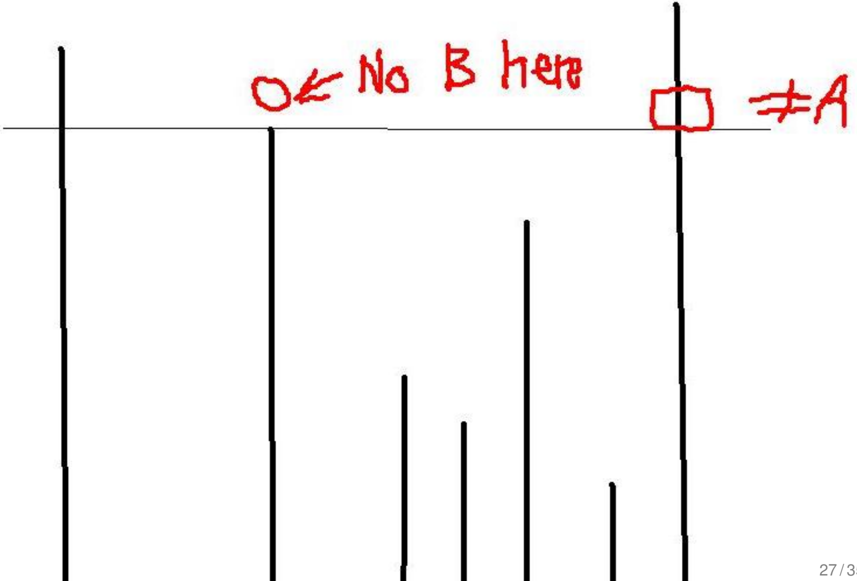
Observation II

Among any lowest k boxes in a column (from floor 1 to floor k), the number of $\#A$-boxes and the number of $\#C$-boxes, denoted by $\#A$ and $\#C$ respectively, satisfy $\#C \leq 3\#A$, namely

$$\boxed{\#A + \#C \leq 4\#A.}$$



Observation III



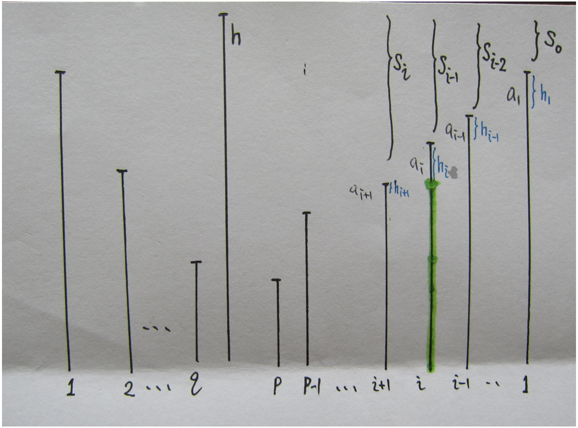


Figure: $h = (\#A + \#C) + \#B \leq 4\#A + a_1 + \cdots + a_p + a'_1 + \cdots + a'_q \leq 4\#A + \frac{h_1 - 1}{4} + \cdots \frac{h_{p-1} - 1}{4} + \frac{h_p}{4} + \frac{h'_1 - 1}{4} + \cdots = 4\#A + \frac{h}{2} - \frac{S_0 + S'_0}{4} - \frac{p+q-1}{2}$

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$h \le 8\#A - 2\times$something $\le 8k - 2\times$something

Observations I and II determine the linear term 8.

General idea for yielding $FF(k) \le h \le 8k - \cdots$: If any of $p + q$, $\frac{h_i}{4} - a_i$, $\frac{h_i}{4} - a_i$, S_0, S'_0 is large, then "something" is large. If $p + q$ is very small, we can try the inequality $a_i \le k$ and get better upper bound of h.

$$h = S_p = S'_q = (\#A+\#C)+\#B \le 4\#A+a_1+\cdots+a_p+a'_1+\cdots+a'_q \le 4\#A+\frac{h_1-1}{4}+\cdots\frac{h_{p-1}-1}{4}+\frac{h_p}{4}+\frac{h'_1-1}{4}+\cdots = 4\#A+\frac{h}{2}-\frac{S_0+S'_0}{4}-\frac{p+q-1}{2}$$

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General idea for yielding $FF(k) \leq h \leq 8k - \cdots$: If any of $p+q$, $\frac{h_i}{4} - a_i$, $\frac{h_i}{4} - a_i$, S_0, S'_0 is large, then "something" is large. If $p+q$ is very small, we can try the inequality $a_i \leq k$ and get better upper bound of h.

Proof of Theorem 3

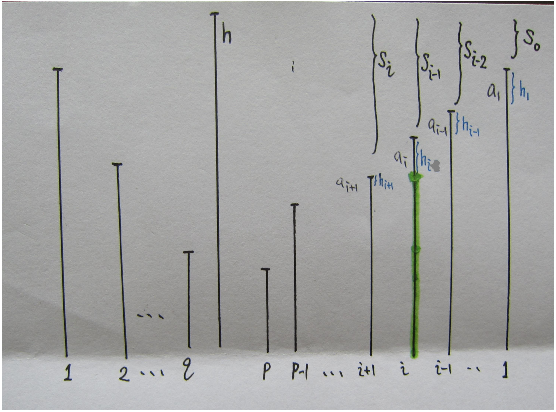

Figure:

$h - S_i = \#B + (\#A + \#C) \leq \frac{h - S_i}{4} + (\frac{h - S_i + h_i + h_{i-1}}{4} - a_{i-1}) + 4(k - a_i)$

Proof of Theorem 3, Contd.

Assume to the contrary that $h > 8k - (2-e)\log_3 k$.

We have $\frac{h_j}{4} - a_{j-1} < C\log_3(k)$, $\frac{h'_j}{4} - a'_{j-1} < C\log_3(k)$, $4k - \frac{h}{2} < C\log_3 k$, $S_0 < C\log_3 k, S'_0 < C\log_3 k$.

We aim to show that

$$p > \log_3(k) - o(\log_3 k), q > \log_3(k) - o(\log_3 k),$$

hence arriving at a contradiction with $\boxed{h \leq 8A - 2^{\frac{p+q-1}{2}} - \cdots}$ when k is sufficiently large.

Proof of Theorem 3, Fin.

$h - S_i \leq \frac{h-S_i}{4} + (\frac{h-S_i+h_i+h_{i-1}}{4} - a_{i-1}) + 4(k - a_i) \Rightarrow 4a_i \leq \frac{h_i}{4} + (\frac{h_{i-1}}{4} - a_{i-1}) + (4k - \frac{h}{2}) + \frac{S_i}{2} < C\log_3 k + \frac{S_i - S_{i-1}}{4} + \frac{S_i}{2} = C\log_3 k + \frac{3S_i - S_{i-1}}{4}$

Substituting the above into $\frac{h_i}{4} - a_i < C\log_3(k)$, we have $C\log_3 k > \frac{h_i}{4} - a_i > \frac{h_i}{4} - C\log_3 k - \frac{3S_i - S_{i-1}}{16} = \frac{S_i}{16} - \frac{3S_{i-1}}{16} - C\log_3 k$ and so $S_i + C\log_3 k < 3(S_{i-1} + C\log_3 k)$.

This gives
$k \leq h = S_p < S_p + C\log_3 k < 3^p(S_0 + C\log_3 k) < 3^p \times 2 \times C\log_3 k$.
Therefore, $\boxed{p > \log_3 k - o(\log_3 k).}$ By symmetry, we have $\boxed{q > \log_3 k - o(\log_3 k).}$

Proof of Theorem 2: $FF(k) \leq 8k-9$

We only need to consider the case of $k \geq 4$. Let $m = FF(k)$ and h be the height of a highest building.

Observation 1: There is a column which contains at least two $\#A$-boxes on its floors $m-3, m-2, m-1$ and m.

This observation gives $h \geq m+4$ as the floors $m+1, \ldots, m+4$ of this same column will be occupied by $\#C$-boxes.

Observation 2: $h \leq 8k-4$. In addition, $h = m+4$ and $h = 8k-4$ cannot hold simultaneously.

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To prove Observation 2, we make use of

$$h \leq 4\#A + \frac{h}{2} - \frac{S_0 + S_0'}{4} - \frac{p+q-1}{2} \tag{1}$$

and distinguish several cases according to the values of (p, q). We also make use of the fact that h is an integer and first

deduce weaker bound for h and then substitute it into the right hand side of Eq. (1) to generate better upper bound of h itself.

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If we try to examine various possible cases for the boxes from floors $m-4$ (or even lower) to floor m in a high column, it should be possible to get better bound for $FF(k)$. But the analysis along this line may be much more complicated and the improvement in constant term may not be so attractive.

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