Ascent Sequences and Semiorders

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What are Ascent Sequences?

Definition: A sequence $a_1, a_2, \ldots, a_n$ is an ascent sequence if $a_1 = 0$ and each $a_i \in [0, \mathrm{asc}(a_1, \ldots, a_{i-1})+1]$.

sequence:	0	1	0	1	0	1	4	3	2	4
ascents:	0	1	1	2	2	3	4	4	4	5

What are Ascent Sequences?

Theorem: (Bousquet-Mélou, Claesson, Dukes, Kitaev '10)

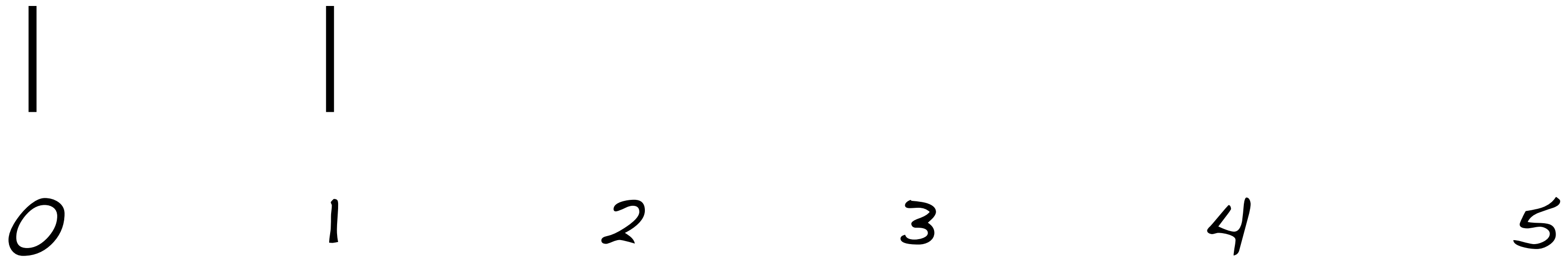
The ascent sequences $a_1, a_2, ..., a_n$ are equinumerous via Ψ with the unlabeled interval orders on n elements.

Bijection Ψ

sequence: 0 1 0 1 0 1 4 3 2 4

$\ell = 0$

$\ell^* = 0$



Bijection Ψ

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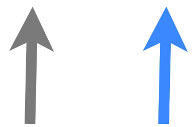
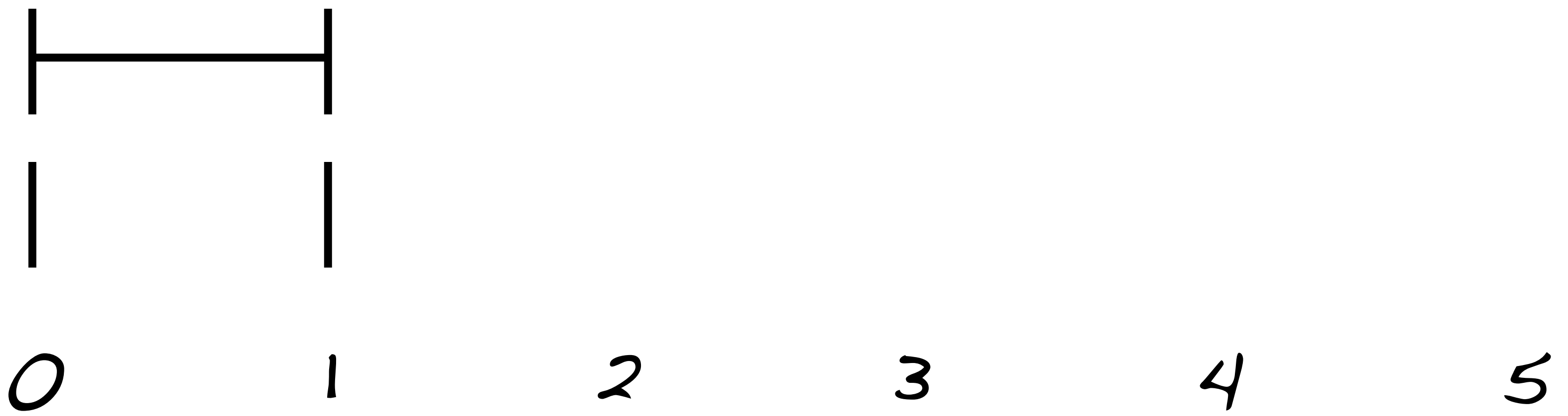


$\ell = 1$

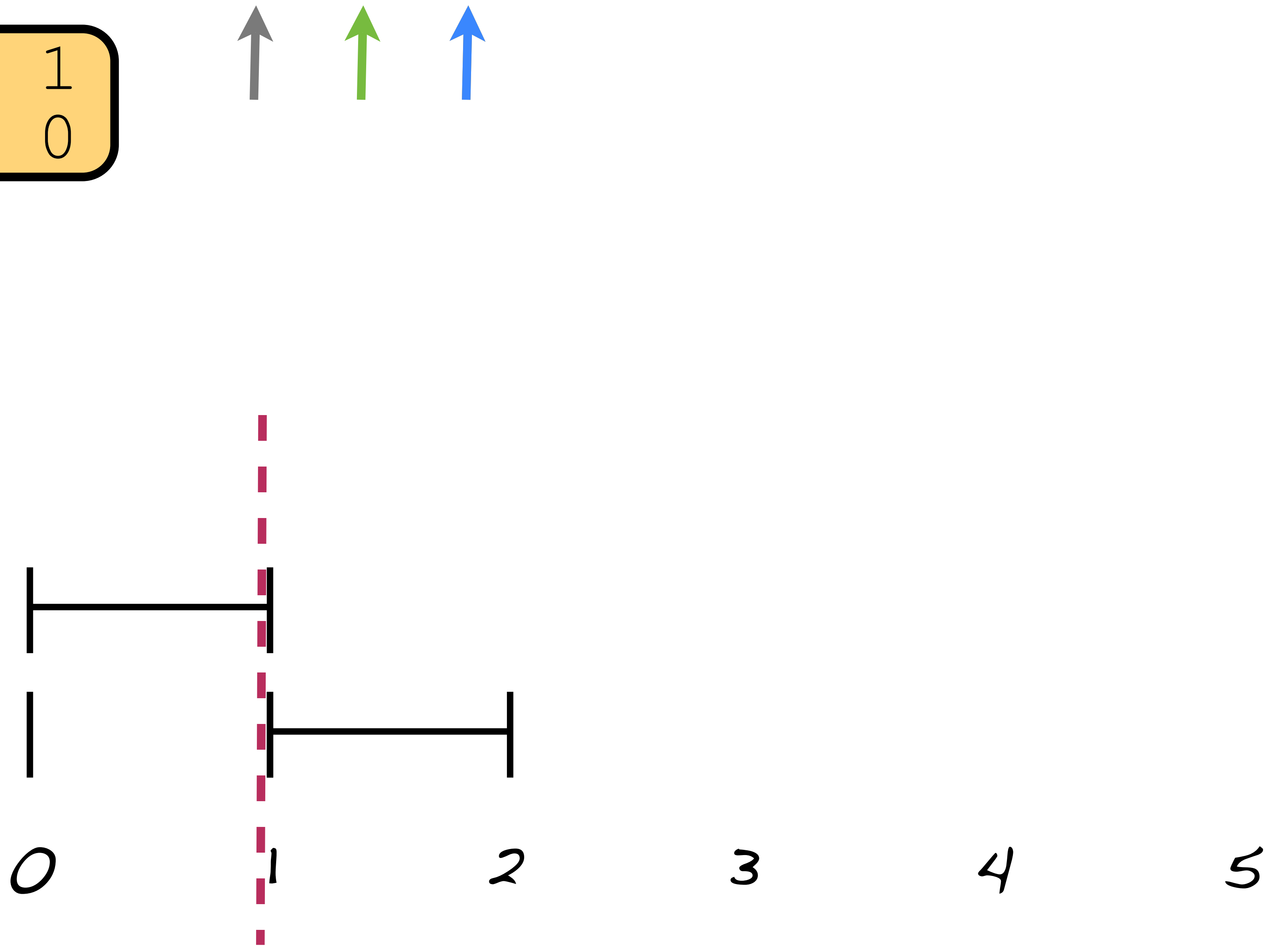
$\ell^* = 1$



Bijection Ψ

Sequence: 0 1 0 1 0 1 4 3 2 4

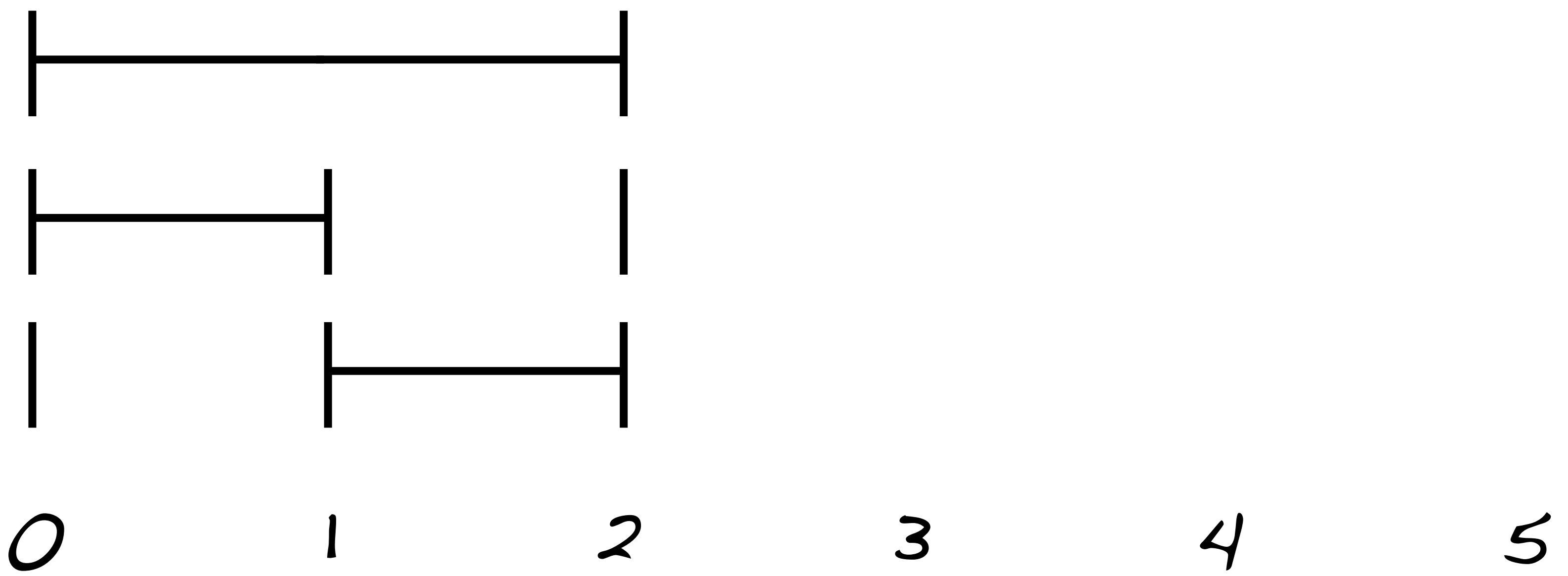
$\ell = 1$
$\ell^* = 0$



Bijection Ψ

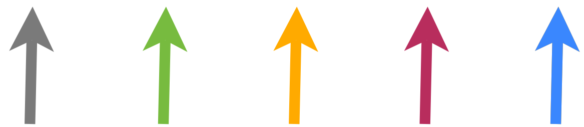
sequence: 0 1 0 1 0 1 4 3 2 4

$\ell = 2$
$\ell^* = 1$



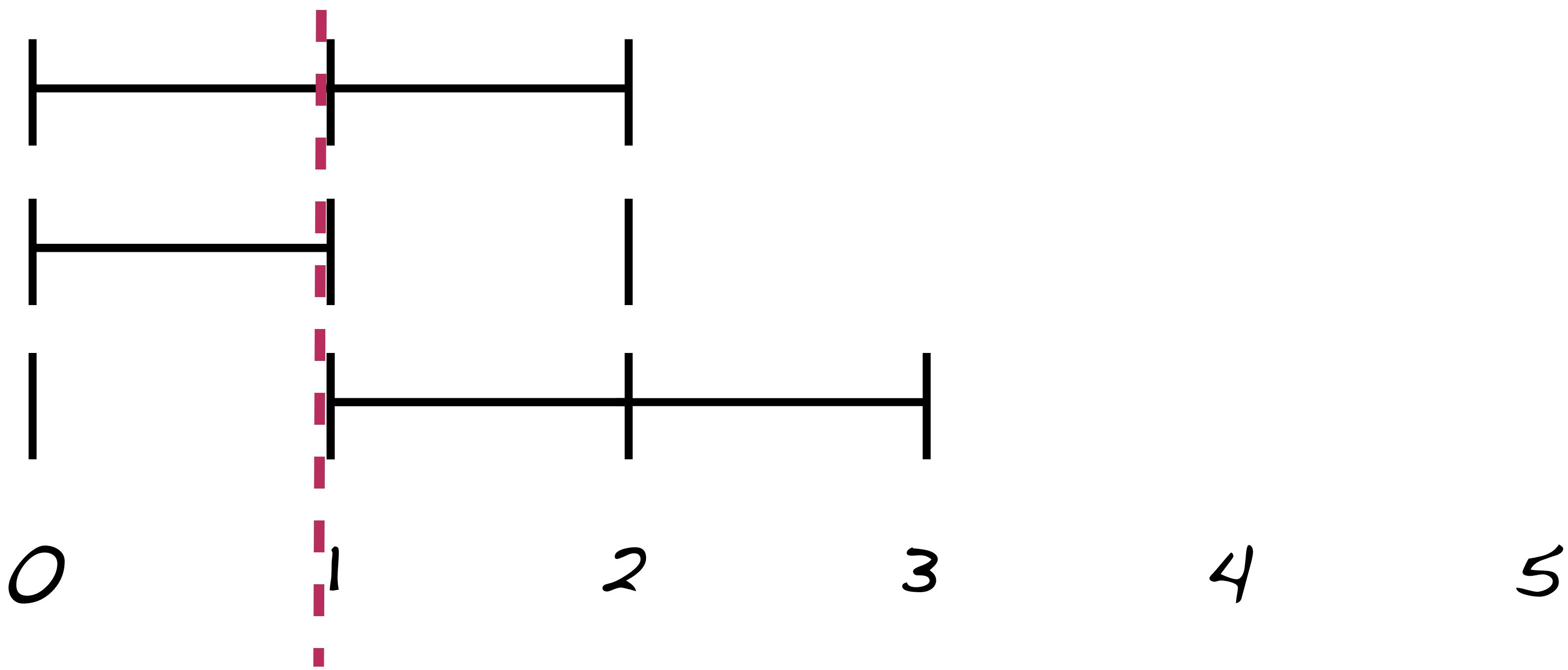
Bijection Ψ

sequence: 0 1 0 1 0 1 4 3 2 4



$\ell = 2$

$\ell^* = 0$



Bijection Ψ

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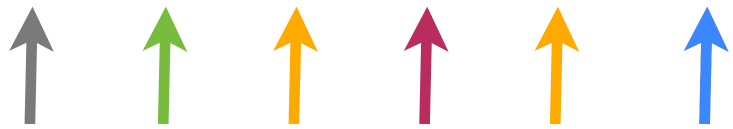
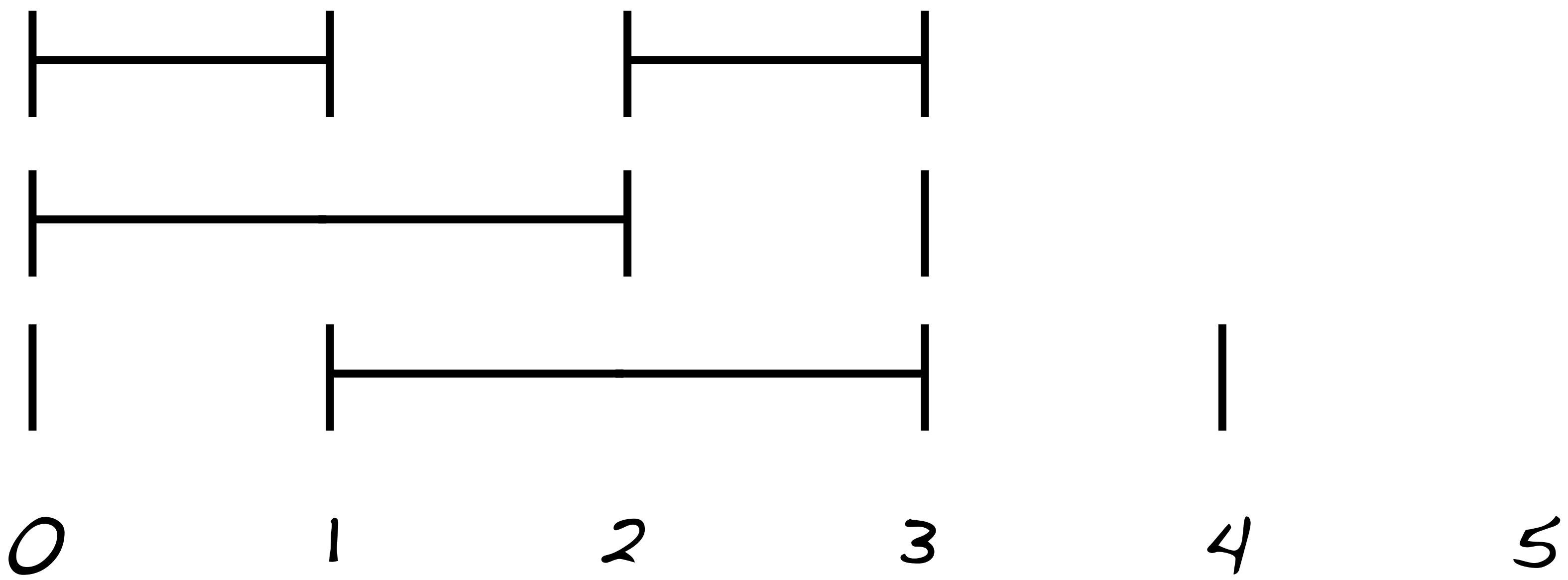


$\ell = 3$

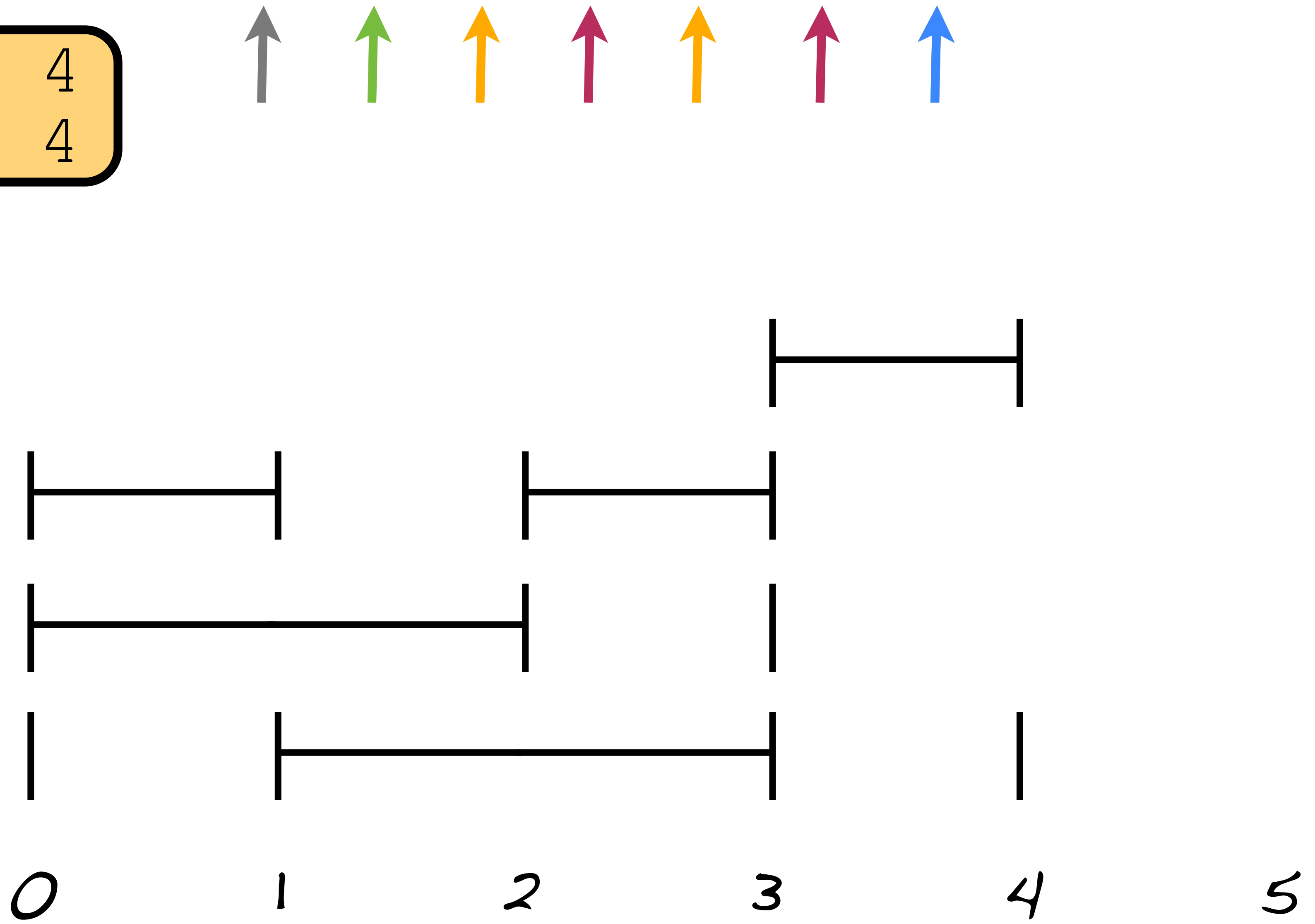
$\ell^* = 1$



Bijection Ψ

sequence: 0 1 0 1 0 1 4 3 2 4

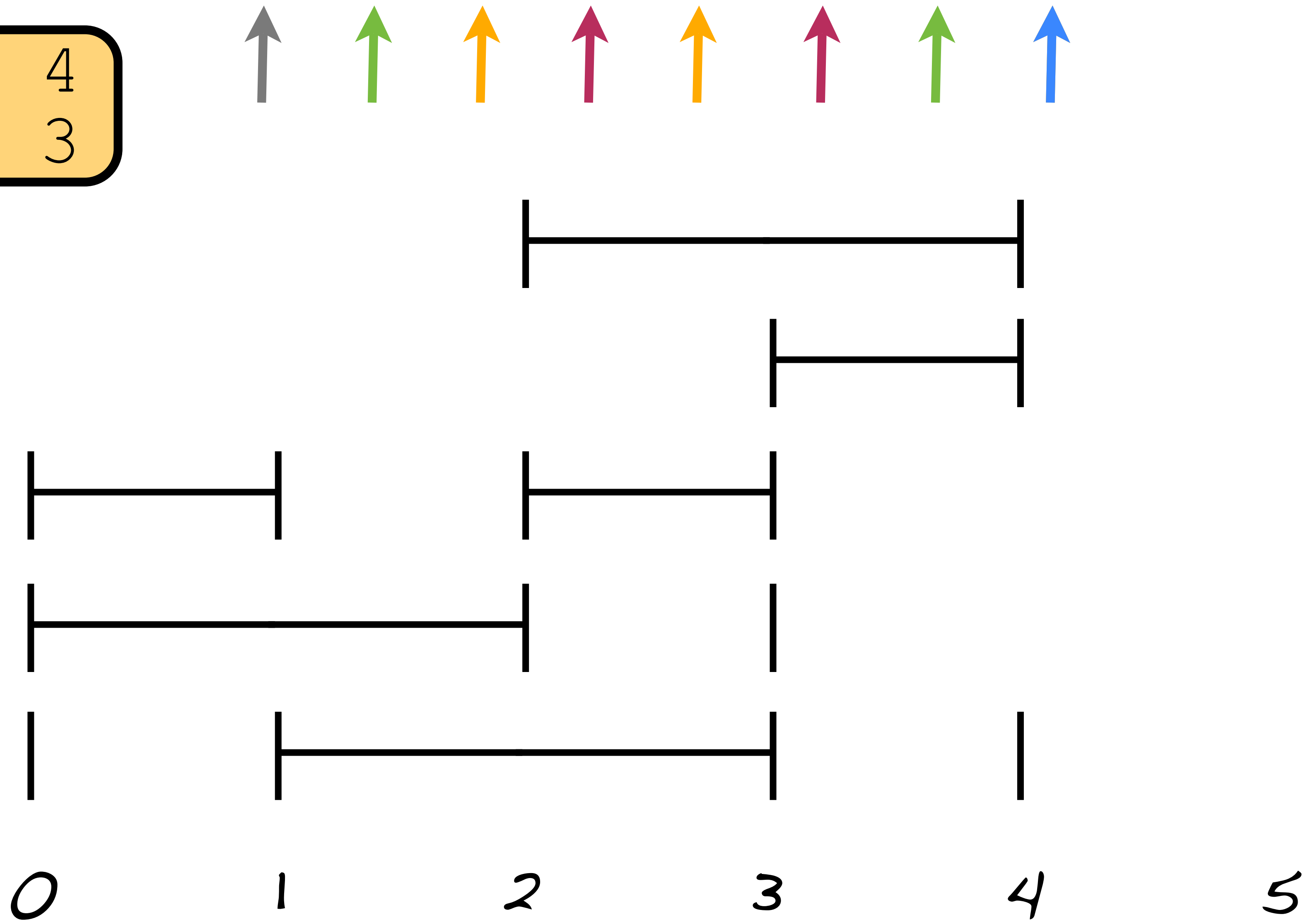
$\ell = 4$
$\ell^* = 4$



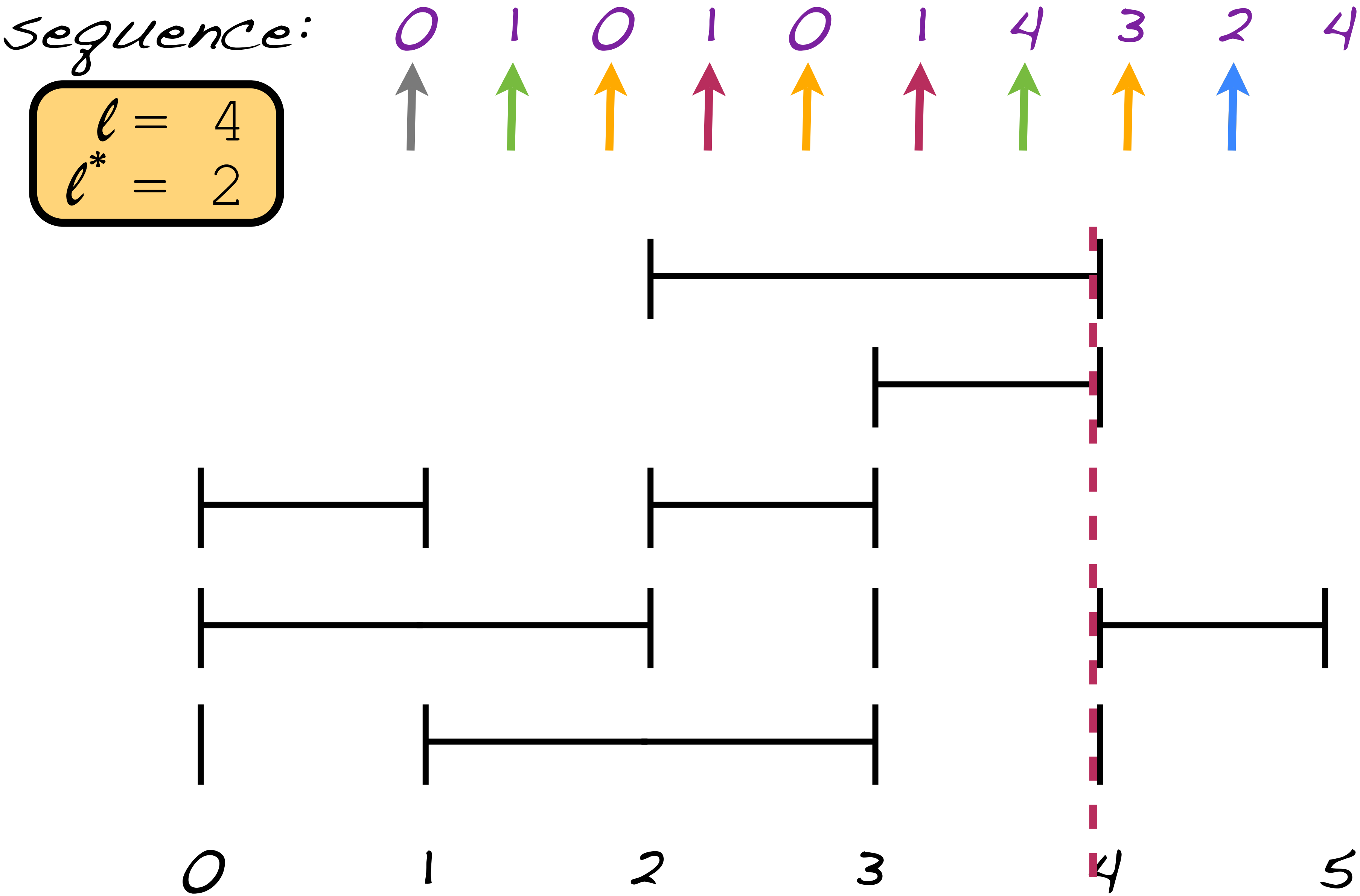
Bijection Ψ

sequence: 0 1 0 1 0 1 4 3 2 4

$\ell = 4$
$\ell^* = 3$



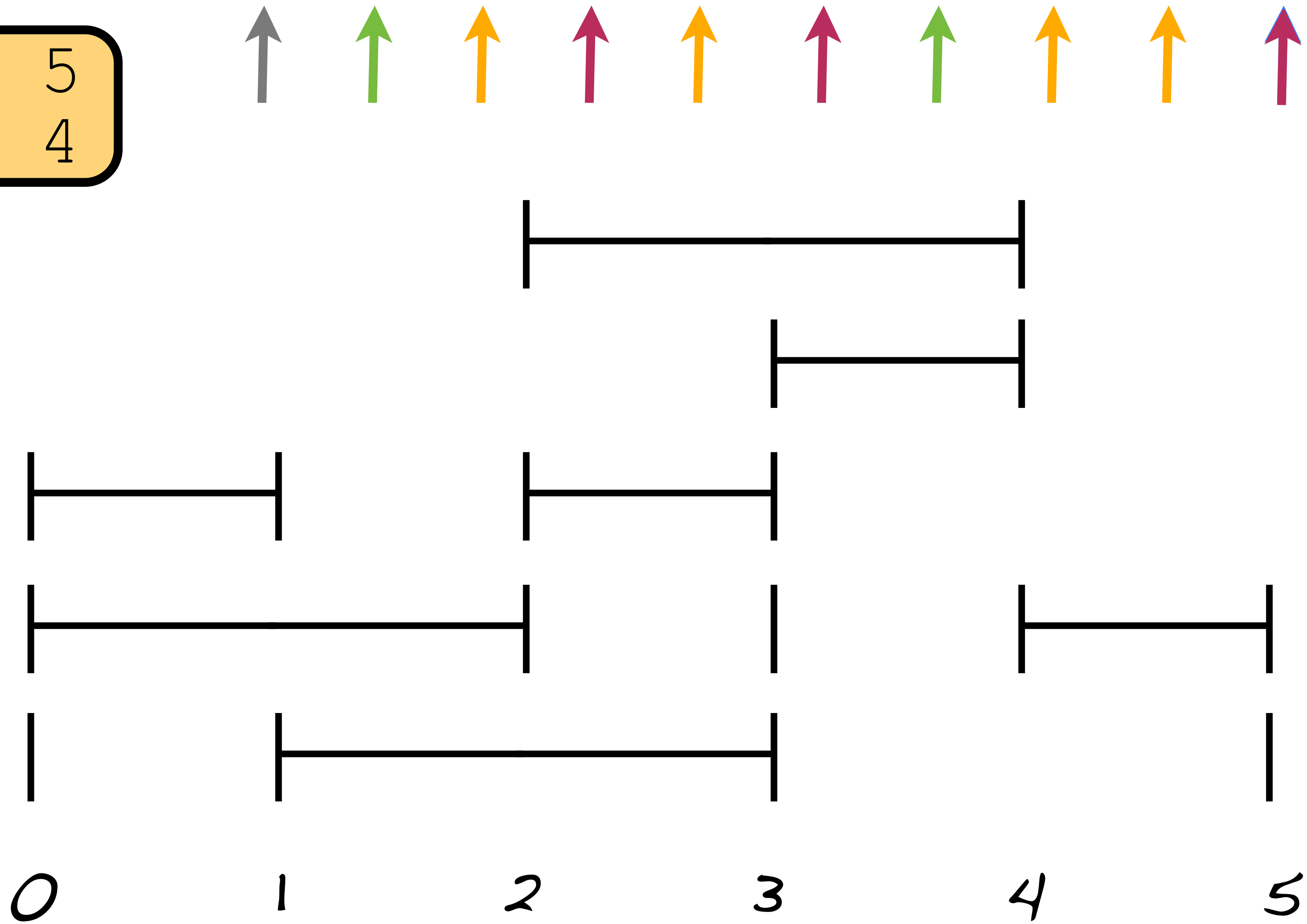
Bijection Ψ



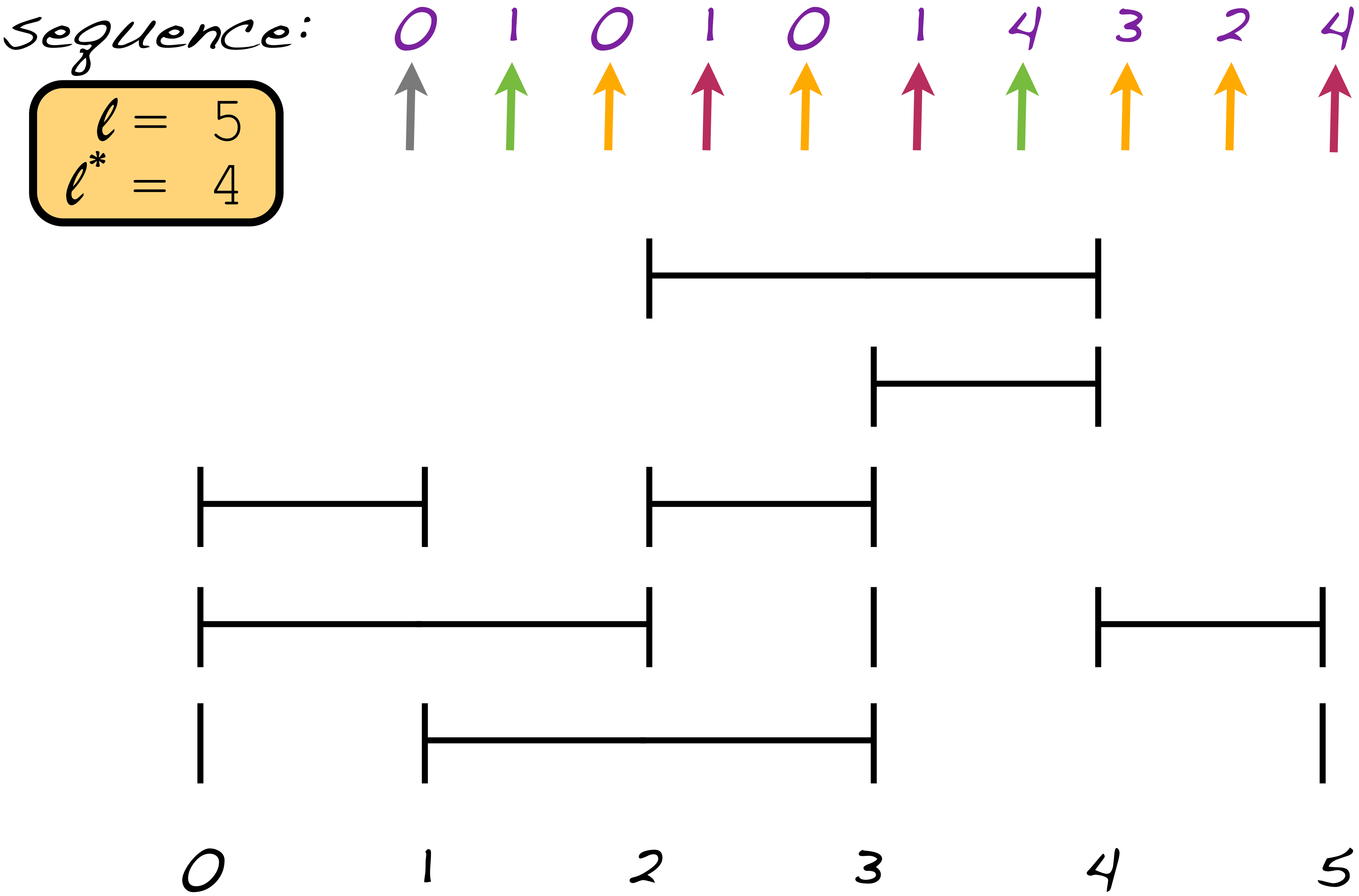
Bijection Ψ

sequence: 0 1 0 1 0 1 4 3 2 4

$\ell = 5$
$\ell^* = 4$



Bijection Ψ



Bijection Ψ

$a_0 = 0$

$\ell = 0$

$\ell^* = 0$

$\mathscr{I} = [0,0]$

$a_i = \ell + 1$

$\ell \leftarrow a_i$

$\ell^* \leftarrow a_i$

$\mathscr{I} \leftarrow [a_i, a_i]$

$a_i \leq \ell^*$

$\ell^* \leftarrow a_i$

$\mathscr{I} \leftarrow [a_i, \ell]$

$\ell^* < a_i \leq \ell$

$\ell \leftarrow \ell + 1$

$\ell^* \leftarrow a_i$

$a_i \leq b < \ell$

$[a, b+1] \leftarrow [a, b]$

$[a+1, b+1] \leftarrow [a, b]$

$[a, a_i] \leftarrow [a, \ell]$

$\mathscr{I} \leftarrow [a_i, \ell+1]$

Enumeration via Ψ

Theorem: (Kitaev, Remmel '11)

Let $G(t,u,v,z,x) = \sum c_{ijklm} t^i u^j v^k z^l x^m$ where c_{ijklm} is the number of unlabelled interval orders in canonical form on i elements with endpoints $\{0,\ldots,j\}$, where $[k,j]$ is the longest maximal interval, there are l minimal elements, and m copies of the interval $[0,0]$,

then $G(t,u,v,z,x) = 1 + zt + (uvzx+z^2)t^2 + (uvzx+u^2v^2zx+uz^2x+uvz^2x^2+z^3)t^3 + \cdots$.

height? *width?* *dimension???*

Restricted Bijection

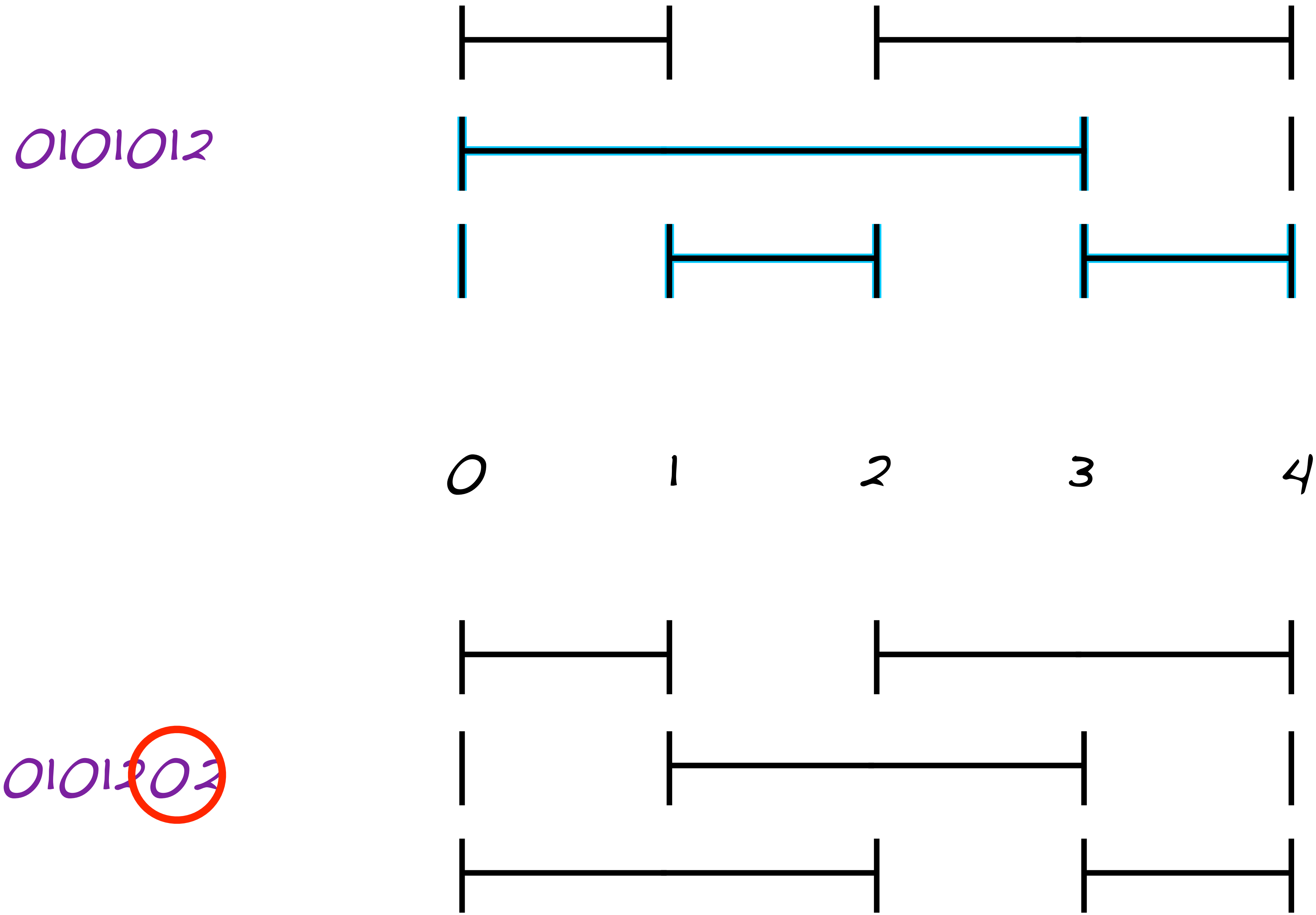
Ascent Sequences $\overset{\Psi}{\longleftrightarrow}$ Interval Orders

$\cup$ $\cup$

Restricted Ascent ??? Sequences? $\overset{\Psi}{\longleftrightarrow}$ Semiorders

Theorem: (Kitaev, Remmel '11) The ascent sequences $a_1, a_2, \ldots, a_n$ where $a_i \geq \max(a_1, \ldots, a_{i-1}) - 1$ (restricted ascent sequences) are equinumerous with semiorders.

Restricted Bijection



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Restricted Bijection

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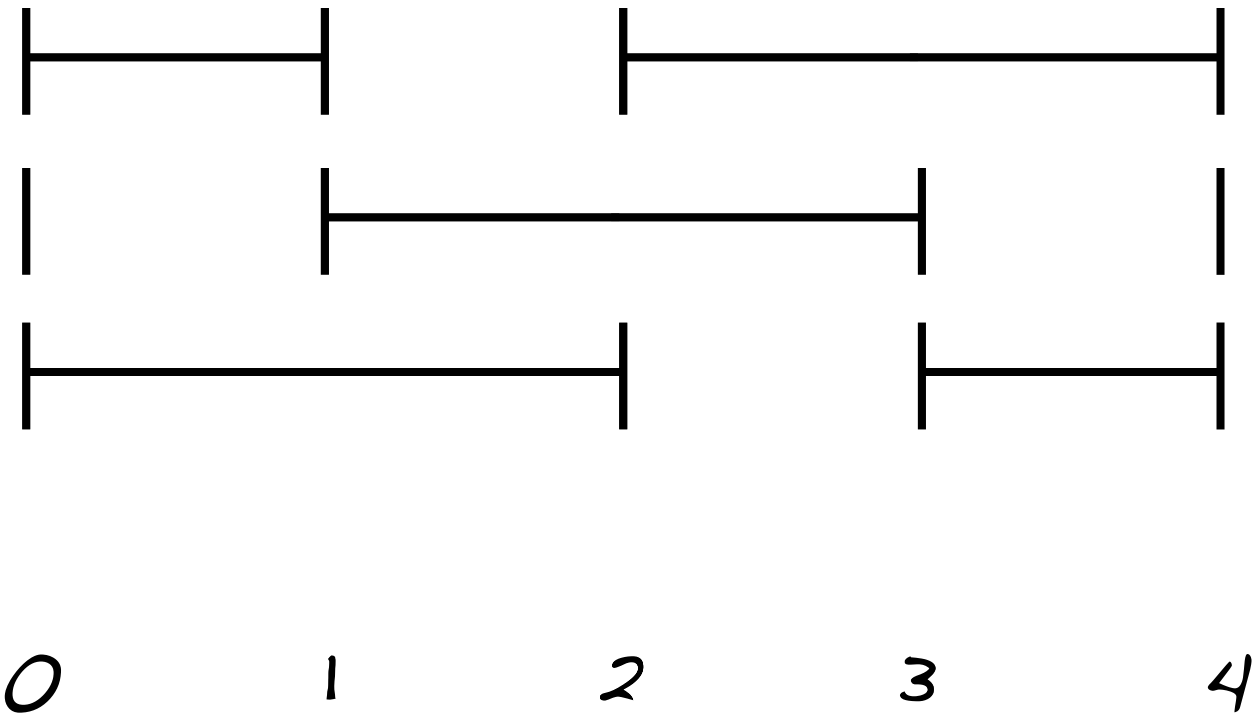

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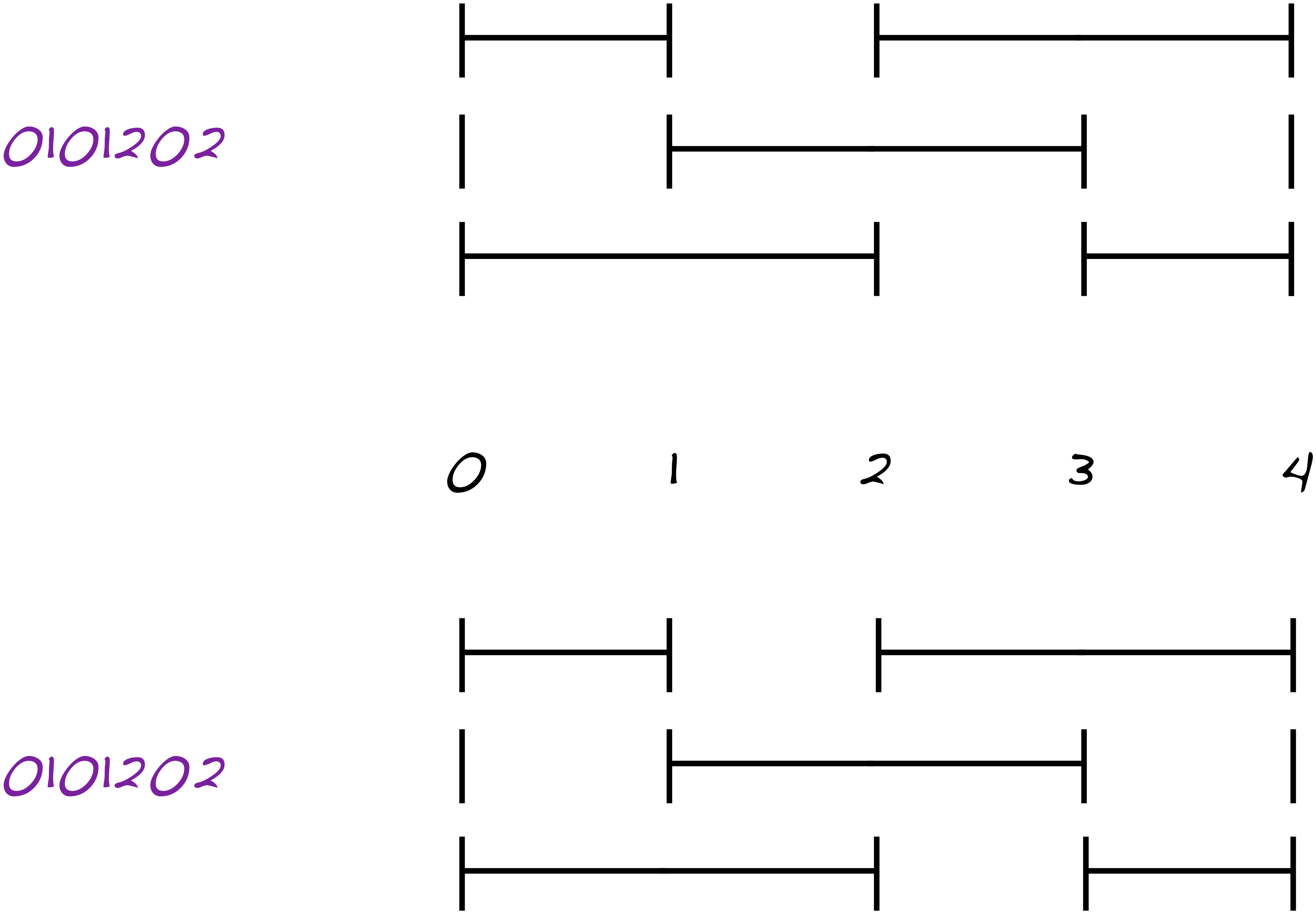
Restricted Bijection



Restricted Bijection

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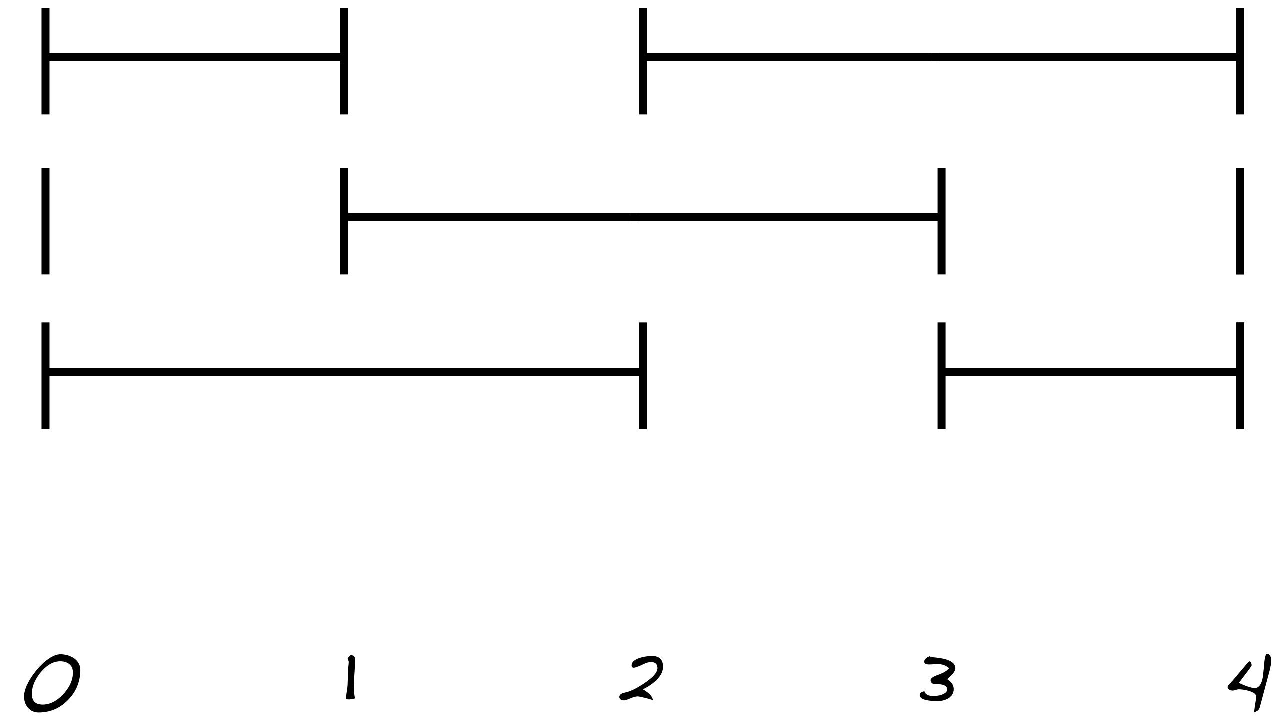

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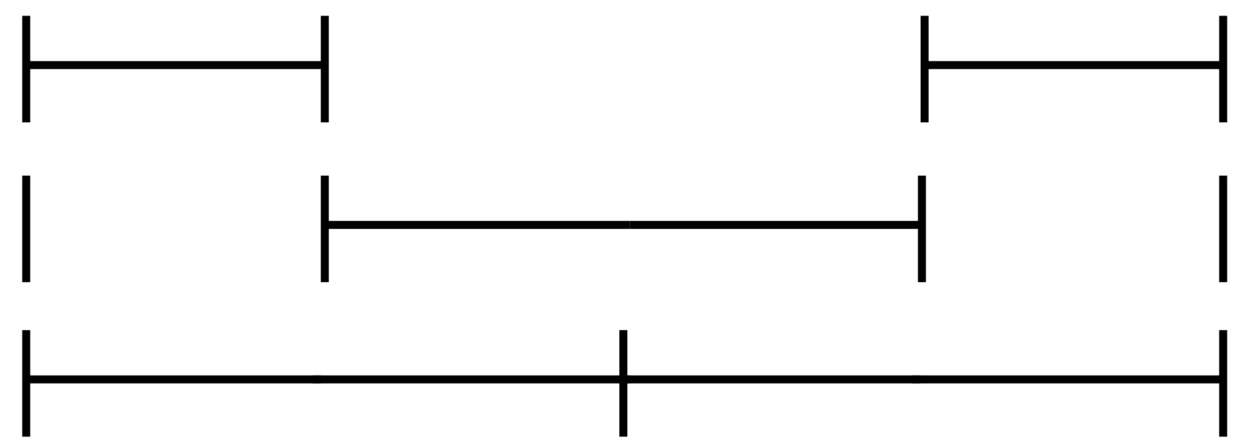
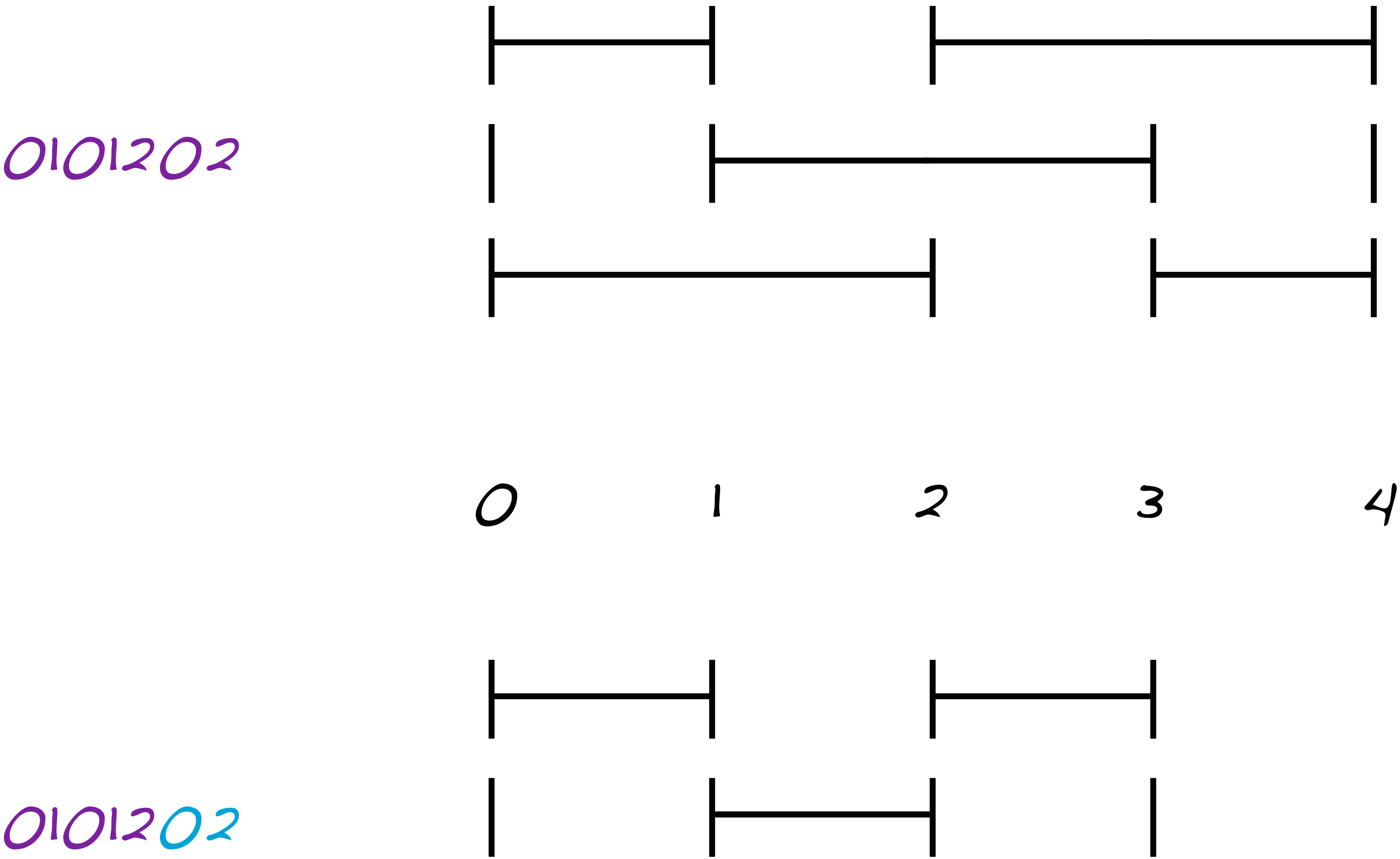
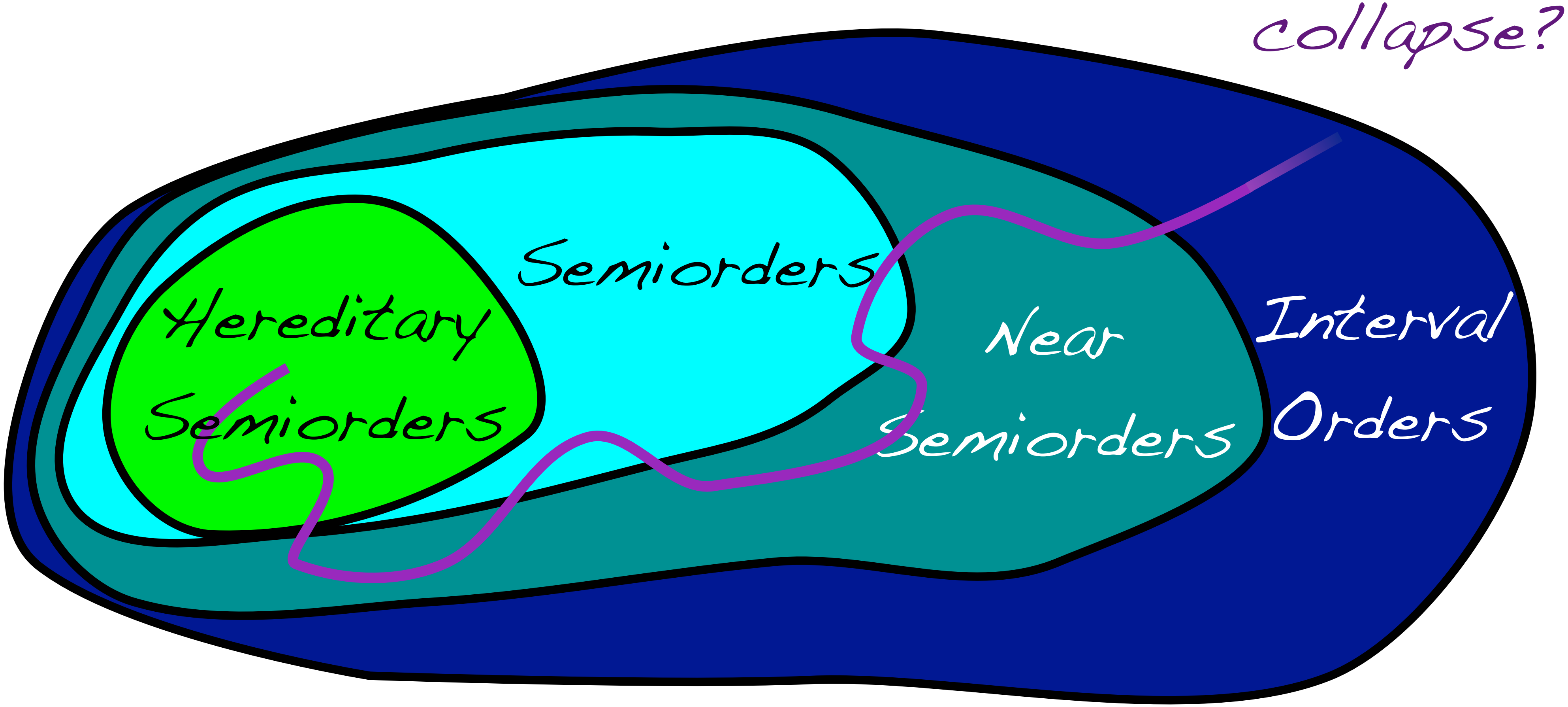


Restricted Bijection

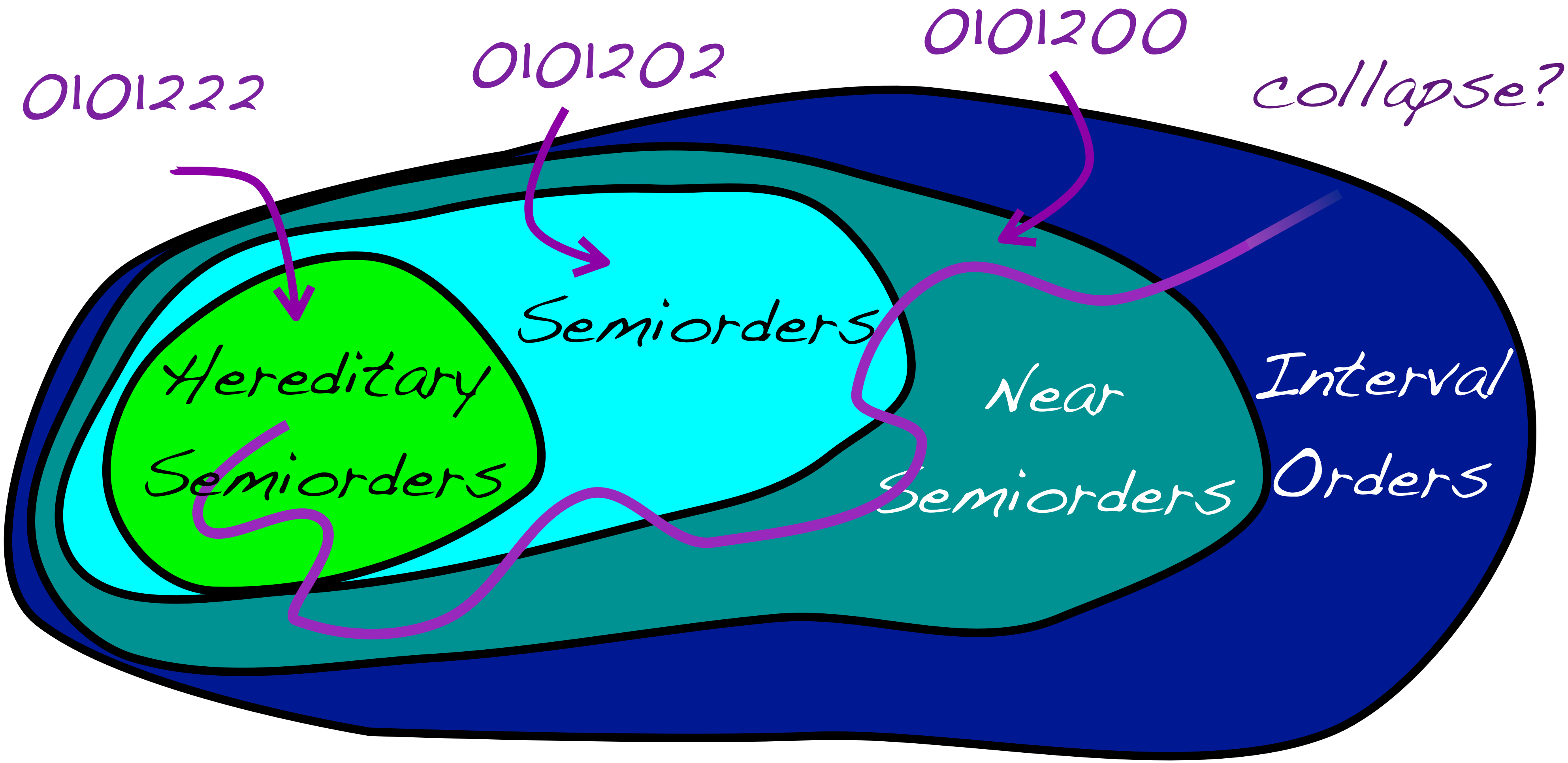


Hereditary Semiorders

Definition: A semiorder S is hereditary if the associated ascent sequence $a_1, a_2, \ldots, a_n$ has the property that $\Psi(a_1, a_2, \ldots, a_k)$ is a semiorder for all k.

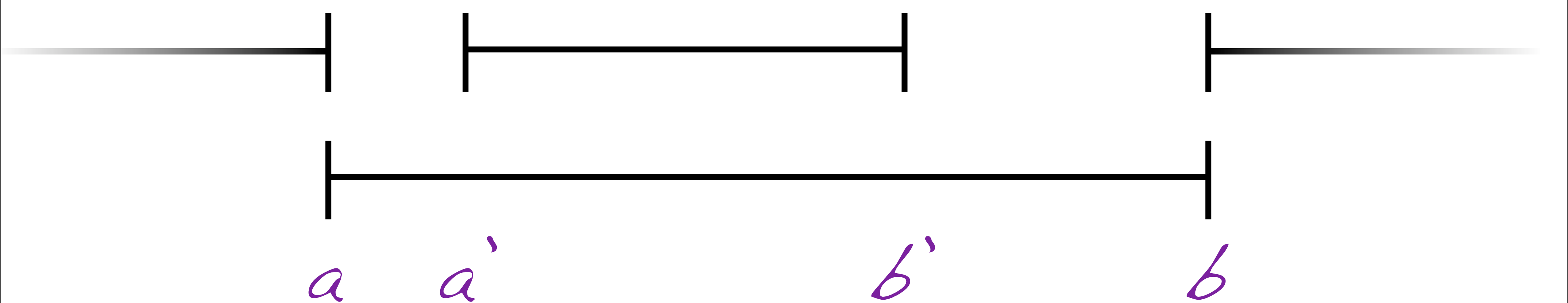


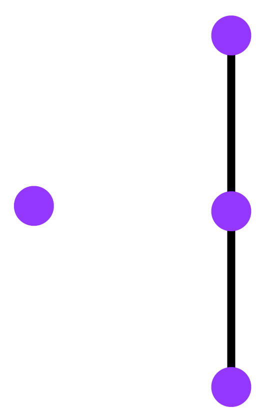
Near Semiorders



Near Semiorders

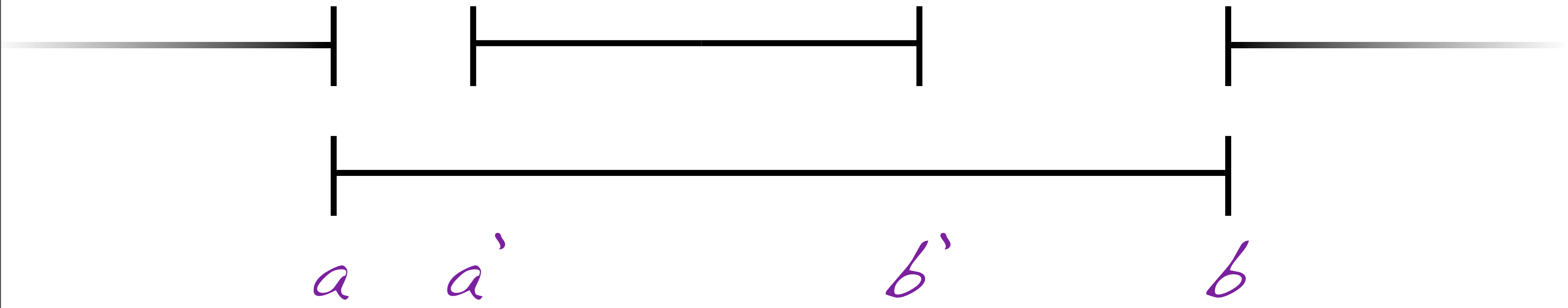
Observation 1: A interval order in cannoncial form I is not a semiorder if and only if there are if there are $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$.





Near Semiorders

Observation 2: A interval order in cannoncial form I is not near semiorder if there is $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$ and either $a' = b'$ or $[a,b]$ is not maximal.



Bijection Ψ

$a_0 = 0$ | $\ell = 0$, $\ell^* = 0$ | $\mathscr{I} = [0,0]$

$a_i = \ell + 1$ | $\ell \leftarrow a_i$, $\ell^* \leftarrow a_i$ | $\mathscr{I} \leftarrow [a_i, a_i]$

$a_i \le \ell^*$ | $\ell^* \leftarrow a_i$ | $\mathscr{I} \leftarrow [a_i, \ell]$

$\ell^* < a_i \le \ell$ | $\ell \leftarrow \ell + 1$, $\ell^* \leftarrow a_i$ |

$a_i \le b < \ell^*$

$[a, b+1] \leftarrow [a, b]$

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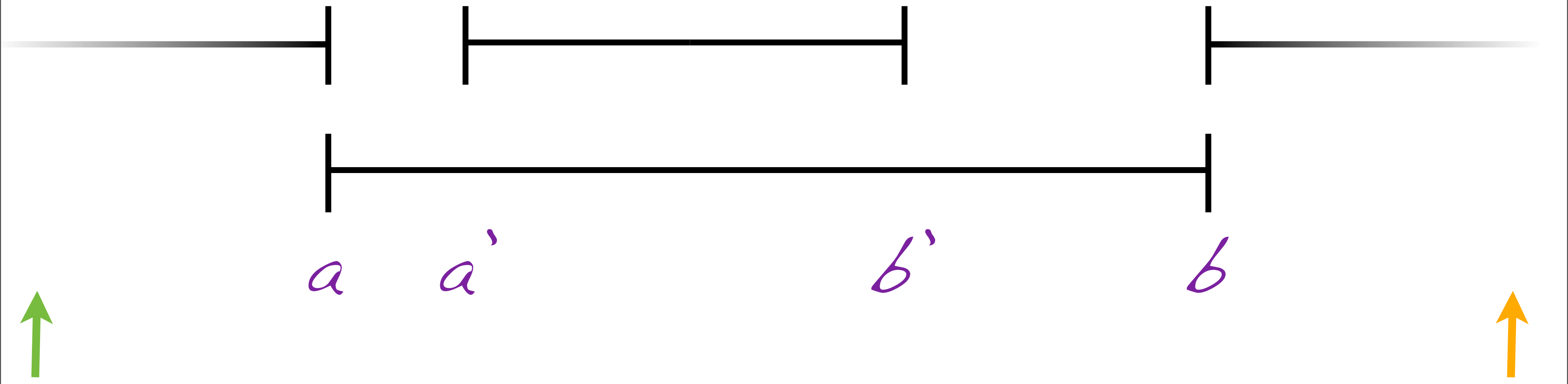

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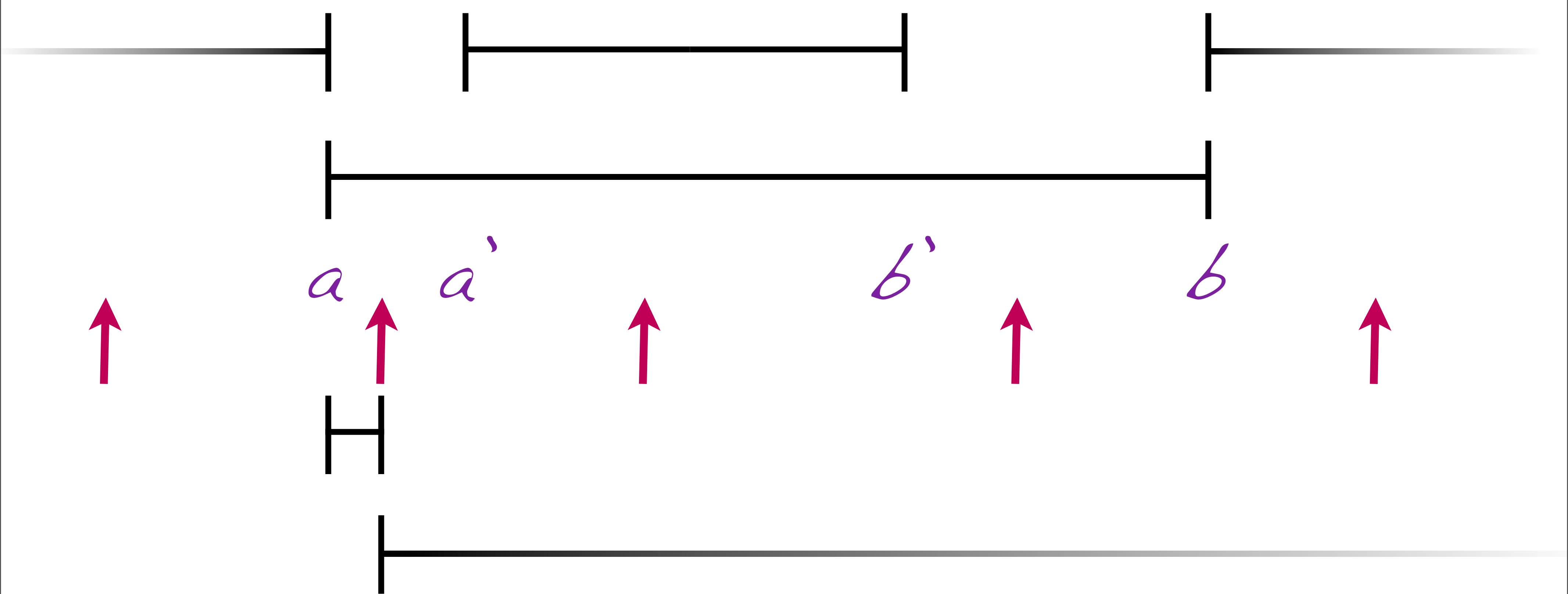

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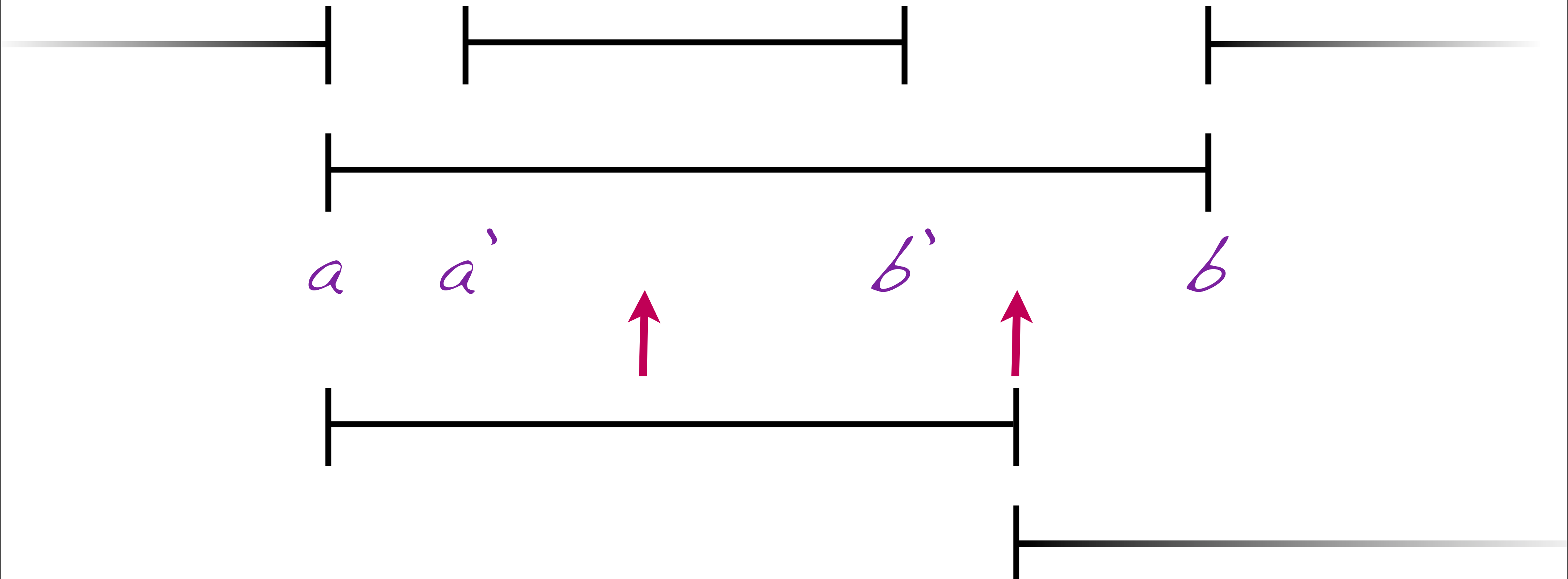

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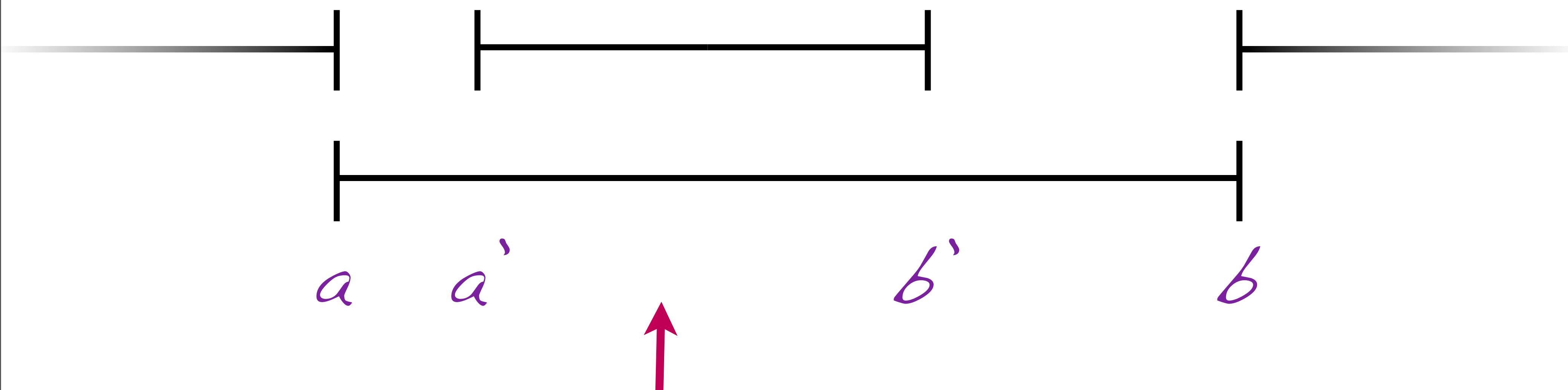
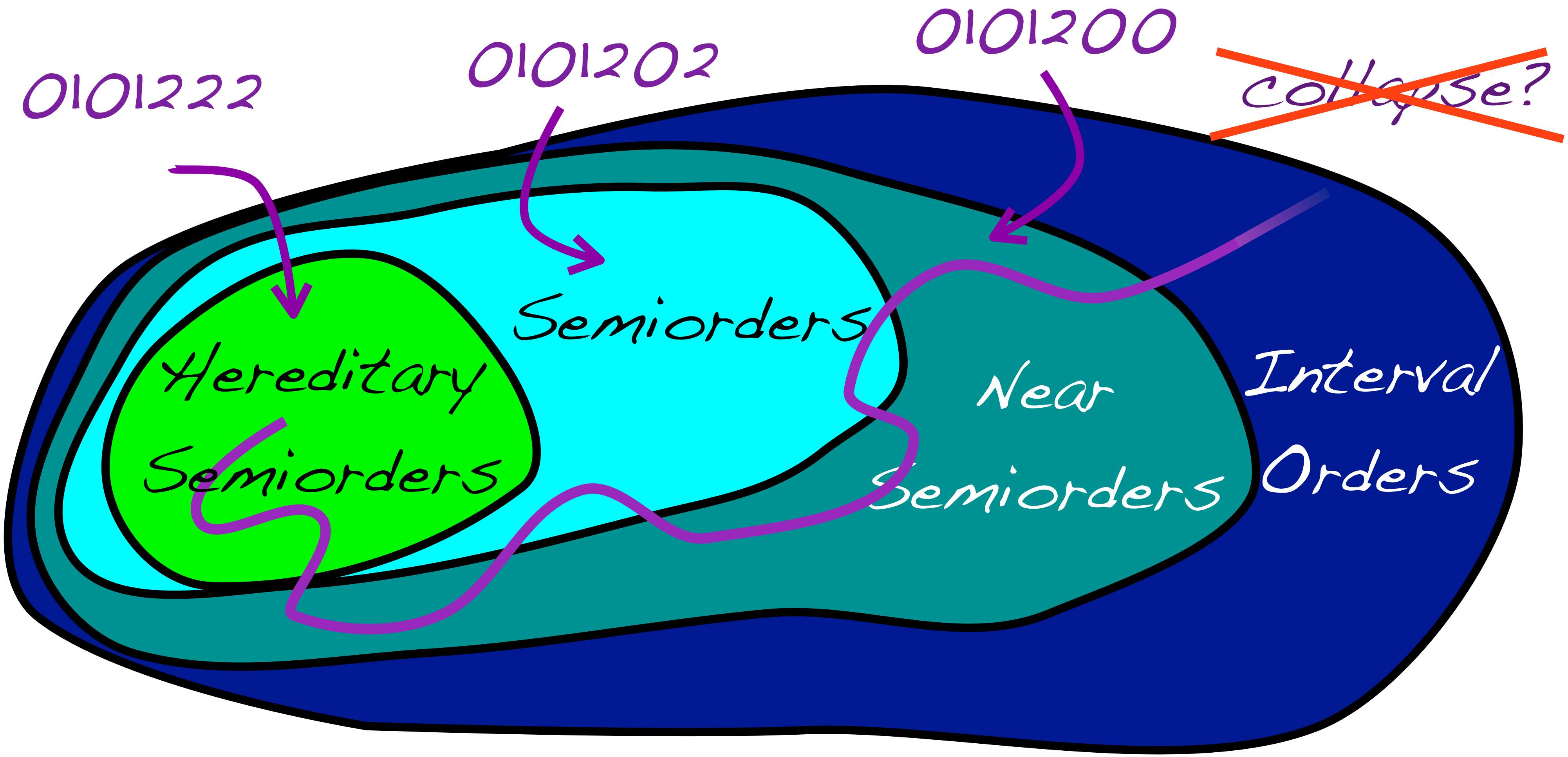
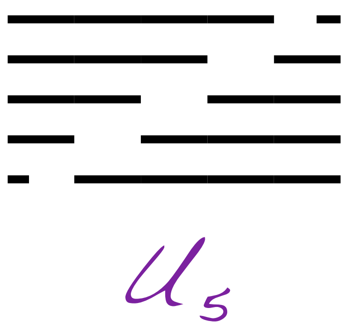


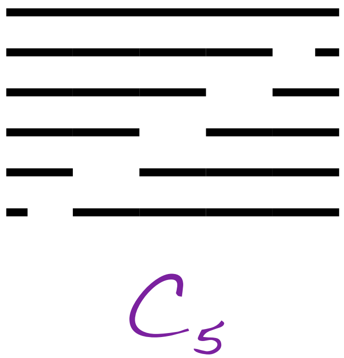
Near Semiorders



Hereditary Semiorders

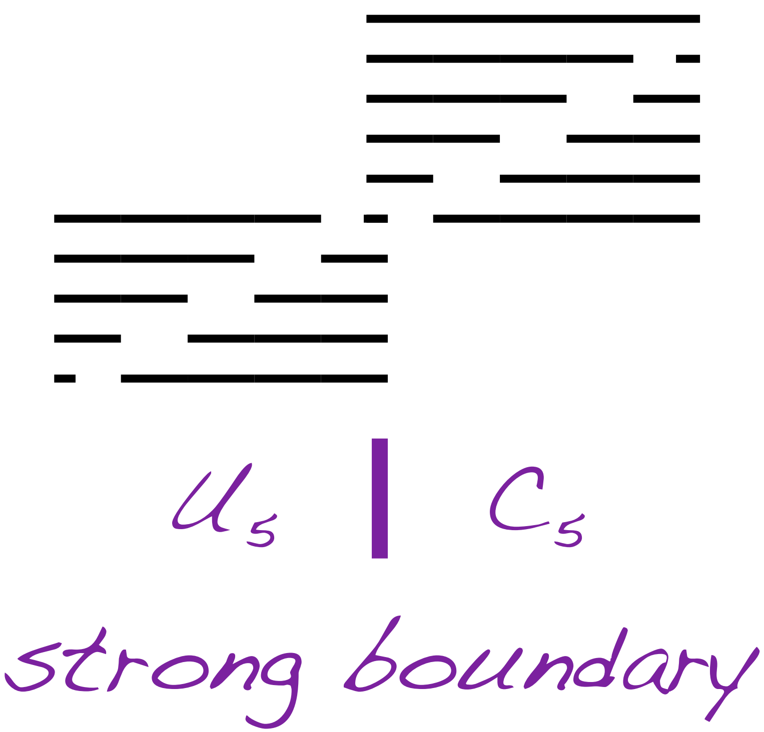
Definition: For an integer x a block of size k is a collection of intervals of the form $[x, x+i]$, $[x+i+1, x+k]$. If the interval $[x, x+k]$ is present, this is a closed block.

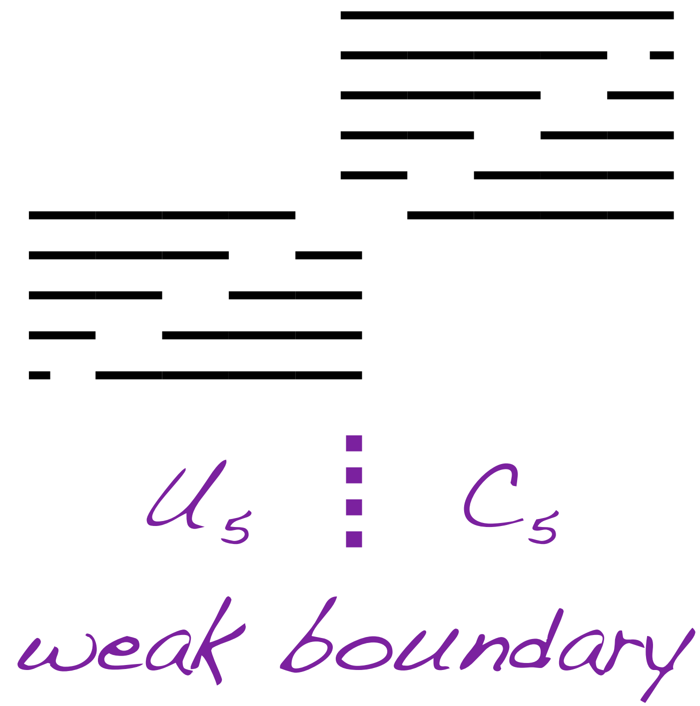




Hereditary Semiorders

Definition: Two blocks B_k and B_i are said to have a strong boundary between them if they share the trivial element. The two blocks are said to share a weak boundary if the trivial element is missing.





Hereditary Semiorders

Theorem: (Remmel and Y. '12+) The collection of hereditary semiorders can be described by an list of blocks together with a collection of strong and weak boundaries*.

$$\mathcal{U}_3 \mid \mathcal{C}_4 \,\vdots\, \mathcal{U}_6 \mid \mathcal{U}_3 \mid \mathcal{C}_8 \mid \mathcal{C}_3 \,\vdots\, \mathcal{U}_2$$

Furthermore the generating function for the number of hereditary semiorders is:

$$\frac{x(5x^6-27x^5+48x^4-46x^3+26x^2-8x+1)}{(1-x)(6x^6-38x^5+65x^4-60x^3+32x^2-9x+1)}$$

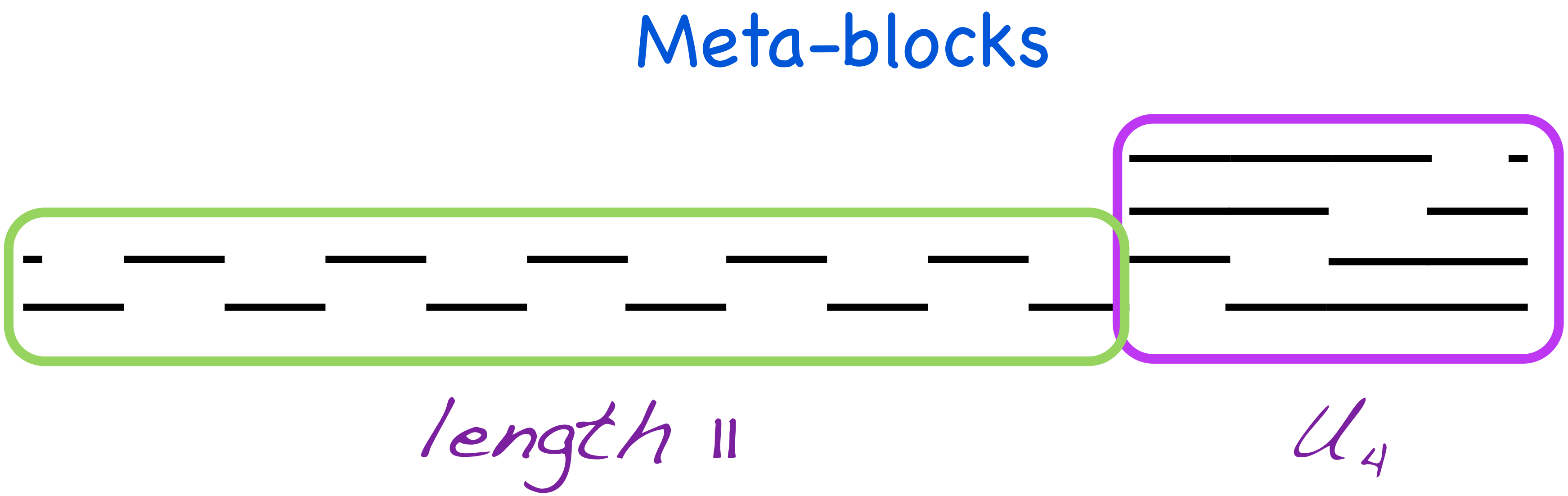
and the number of hereditary semiorders on n elements is approximately

3.3704^n.

Hereditary Semiorders

Hereditary Semiorders

Corollary: (Remmel and Y. '12+) The collection of hereditary semiorders can be described by an list of meta-blocks* together with a collection of strong and weak boundaries*.



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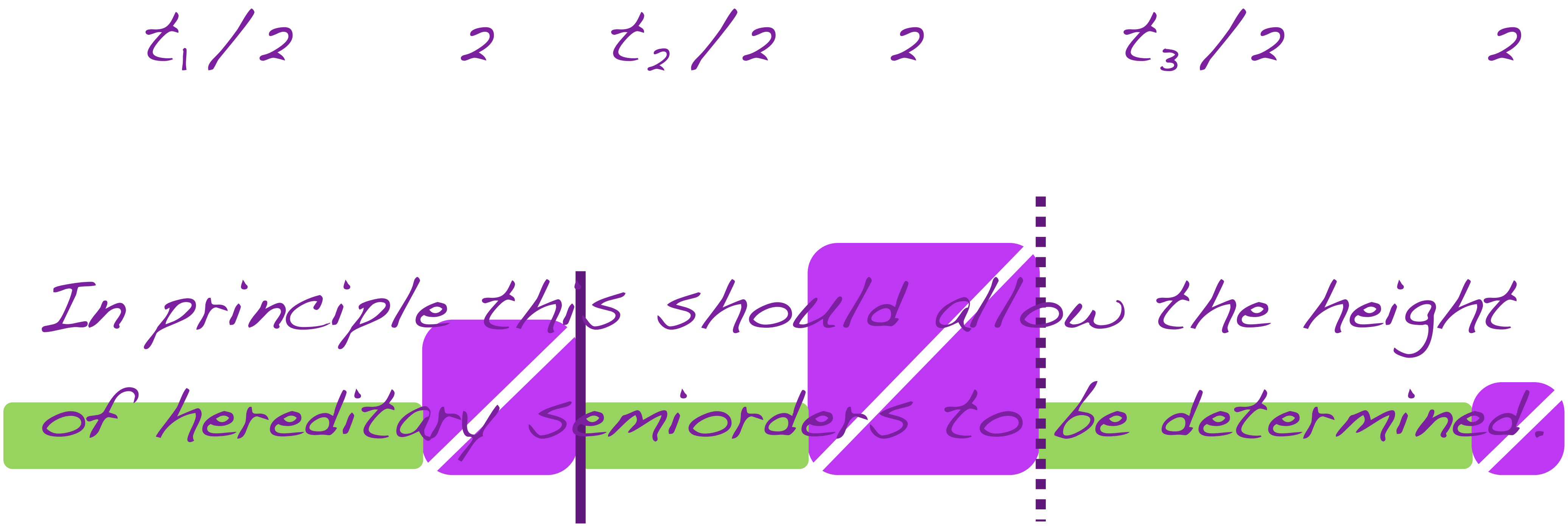
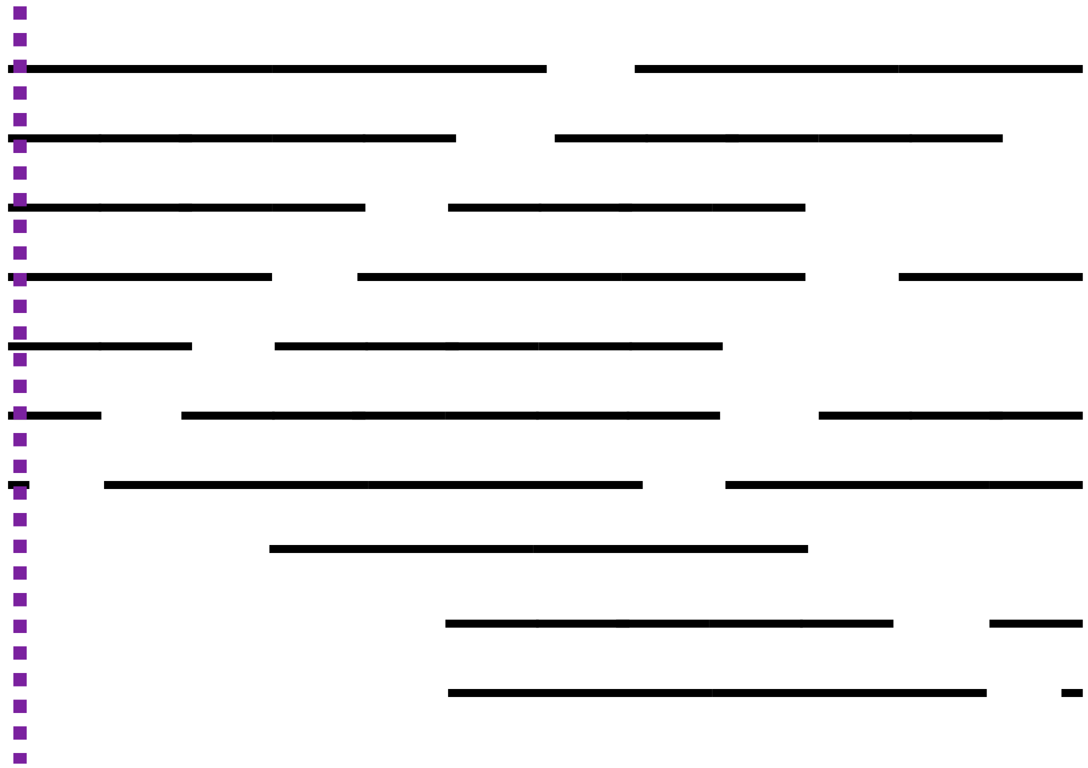
Hereditary Semiorders



In principle this should allow the height of hereditary semiorders to be determined.

Remaining Semiorders



Open Questions

height?

width?

dimension?

maximal elements?

Is Ψ even the right bijection?