Ascent Sequences and Semiorders

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What are Ascent Sequences?

**Definition:** A sequence $a_1, a_2, \ldots, a_n$ is an ascent sequence if $a_1 = 0$ and each

$$a_i \in [0, \text{asc}(a_1, \ldots, a_{i-1})+1].$$

sequence: 0 1 0 1 0 1 4 3 2 4

ascents: 0 1 1 1 2 2 3 4 4 4 4 5
What are Ascent Sequences?

**Theorem:** (Bousquet-Mélou, Claesson, Dukes, Kitaev ‘10)
The ascent sequences $a_1, a_2, \ldots, a_n$ are equinumerous via $\Psi$ with the unlabeled interval orders on $n$ elements.
Bijection $\Psi$

**Sequence:**

\[
\begin{align*}
l &= 0 \\
l^* &= 0
\end{align*}
\]
Bijection $\Psi$

Sequence: $0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 1$
$l^* = 1$
Bijection $\Psi$

**Sequence:** $0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 1$

$l^* = 0$
Bijection $\Psi$

**Sequence:** 0 1 0 1 0 1 4 3 2 4

$l = 2$

$l^* = 1$
Bijection $\Psi$

Sequence: $0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 2$

$l^* = 0$
Bijection $\Psi$

Sequence: $0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 3$

$l^* = 1$
Bijection $\Psi$

Sequence: $0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 4$

$l^* = 4$
Bijection $\Psi$

Sequence: 0 1 0 1 0 1 4 3 2 4

$l = 4$

$l^* = 3$
Bijection $\Psi$

**Sequence:**

```
0 1 0 1 0 1 4 3 2 4
```

```
l = 4
l* = 2
```
Bijection $\Psi$

Sequence:

$l = 5$

$l^* = 4$
**Bijection $\Psi$**

**Sequence:**

$0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 4 \ 3 \ 2 \ 4$

$l = 5$

$l^* = 4$
Bijection $\Psi$

\[ a_0 = 0 \quad l = 0 \quad l^* = 0 \quad I = [0,0] \]

\[ a_i = l + 1 \quad l \leftarrow a_i \quad l^* \leftarrow a_i \quad I \leftarrow [a_i, a_i] \]

\[ a_i \leq l^* \quad l^* \leftarrow a_i \quad I \leftarrow [a_i, l] \]

\[ l^* < a_i \leq l \quad l \leftarrow l + 1 \quad l^* \leftarrow a_i \quad a_i \leq b < l \]

\[ [a, b+1] \leftarrow [a, b] \quad [a+1, b+1] \leftarrow [a, b] \quad [a, a_i] \leftarrow [a, l] \quad I \leftarrow [a_i, l+1] \]
Enumeration via $\Psi$

**Theorem:** (Kitaev, Remmel ‘11)

Let $G(t,u,v,z,x) = \sum c_{ijklm} t^i u^j v^k z^l x^m$ where $c_{ijklm}$ is the number of unlabelled interval orders in canonical form on $i$ elements with endpoints $\{0, \ldots, j\}$, where $[k,j]$ is the longest maximal interval, there are $l$ minimal elements, and $m$ copies of the interval $[0,0]$.

Then $G(t,u,v,z,x) = 1 + zt + (uvzx+z^2)t^2 + (uvzx+u^2v^2zx+uz^2x+uvz^2x^2+z^3)t^3 + \ldots$.
**Theorem**: (Kitaev, Remmel ‘11) The ascent sequences $a_1, a_2, \ldots, a_n$ where $a_i \geq \max(a_1, \ldots, a_{i-1}) - 1$ (restricted ascent sequences) are equinumerous with semiorders.
Restricted Bijection

0101012

0 1 2 3 4

0101202
**Theorem**: (Kitaev, Remmel ‘11) The ascent sequences $a_1, a_2, \ldots, a_n$ where $a_i \geq \max(a_1, \ldots, a_{i-1}) - 1$ (restricted ascent sequences) are equinumerous with semiorders.
Restricted Bijection

0101202

0 1 2 3 4
Restricted Bijection

0101202

0 1 2 3 4

0101202
Restricted Bijection

0101202

0  1  2  3  4

0101202
Restricted Bijection

0101202

0 1 2 3 4

0101202
**Definition:** A semiorder $S$ is hereditary if the associated ascent sequence $a_1, a_2, \ldots, a_n$ has the property that $\Psi(a_1, a_2, \ldots, a_k)$ is a semiorder for all $k$.
Near Semiorders

**Observation 1**: A interval order in cannoncional form $I$ is not a semiorder if and only if there are $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$. 
Near Semiorders

Observation 2: A interval order in cannonncial form $I$ is not near semiorder if there is $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$ and either $a' = b'$ or $[a,b]$ is not maximal.
Bijection $\Psi$

\[
\begin{align*}
  a_0 &= 0 & l &= 0 & \mathcal{I} &= [0, 0] \\
  l^* &= 0
\end{align*}
\]

\[
\begin{align*}
  a_i &= l + 1 & l &\leftarrow a_i & \mathcal{I} &\leftarrow [a_i, a_i] \\
  l^* &= a_i
\end{align*}
\]

\[
\begin{align*}
  a_i &\leq l^* & l^* &\leftarrow a_i & \mathcal{I} &\leftarrow [a_i, l] \\
  l &\leftarrow l^* \\
  l^* &\leftarrow a_i
\end{align*}
\]

\[
\begin{align*}
  l^* &< a_i \leq l & l &\leftarrow l + 1 \\
  l^* &\leftarrow a_i
\end{align*}
\]

\[
\begin{align*}
  a_i &\leq b < l^* & [a, b+1] &\leftarrow [a, b] \\
  [a+1, b+1] &\leftarrow [a, b] \\
  [a, a_i] &\leftarrow [a, l^*] \\
  \mathcal{I} &\leftarrow [a_i, l+1]
\end{align*}
\]
Observation 2: A interval order in canonical form $I$ is not near semiorder if there is $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$ and either $a' = b'$ or $[a,b]$ is not maximal.
### Bijective Mapping $\Psi$

- **Case 1:** $a_0 = 0$, $l = 0$, $l^* = 0$, $\mathcal{I} = [0,0]$

- **Case 2:** $a_i = l + 1$, $l \leftarrow a_i$, $l^* \leftarrow a_i$, $\mathcal{I} \leftarrow [a_i, a_i]$

- **Case 3:** $a_i \leq l^*$, $l^* \leftarrow a_i$, $\mathcal{I} \leftarrow [a_i, l]$

- **Case 4:** $l^* < a_i \leq l$, $l \leftarrow l + 1$, $l^* \leftarrow a_i$, $a_i \leq b < l^*$, 
  $[a, b+1] \leftarrow [a, b]$, 
  $[a+1, b+1] \leftarrow [a, b]$, 
  $[a, a_i] \leftarrow [a, l^*]$, 
  $\mathcal{I} \leftarrow [a_i, l+1]$
Near Semiorders

Observation 2: A interval order in canonical form $I$ is not near semiorder if there is $a < a' \leq b' < b$ such that $[a,b], [a',b'] \in I$ and either $a' = b'$ or $[a,b]$ is not maximal.
Near Semiorders

**Observation 2:** A interval order in canononical form \( I \) is not near semiorder if there is \( a < a' \leq b' < b \) such that \( [a,b], [a',b'] \in I \) and either \( a' = b' \) or \( [a,b] \) is not maximal.
Near Semiorders

**Observation 2:** A interval order in cannoncial form \( I \) is not near semiorder if there is \( a < a' \leq b' < b \) such that \([a,b], [a',b'] \in I\) and either \( a' = b' \) or \([a,b]\) is not maximal.
Near Semiorders

Interval Orders

Hereditary Semiorders

Semiorders

Near Semiorders

collapse?
Hereditary Semiorders

**Definition:** For an integer $x$ a block of size $k$ is a collection of intervals of the form $[x,x+i], [x+i+1,x+k]$. If the interval $[x,x+k]$ is present, this is a closed block.
**Hereditary Semiorders**

**Definition:** Two blocks $B_k$ and $B_i$ are said to have a strong boundary between them if they share the trivial element. The two blocks are said to share a weak boundary if the trivial element is missing.
Hereditary Semiorders

**Theorem:** (Remmel and Y. ‘12+) The collection of hereditary semiorders can be described by an list of blocks together with a collection of strong and weak boundaries*.

\[ U_3 \, | \, C_4 \, | \, U_6 \, | \, U_3 \, | \, C_8 \, | \, C_3 \, | \, U_2 \]

Furthermore the generating function for the number of hereditary semiorders is:

\[
\frac{x(5x^6-27x^5+48x^4-46x^3+26x^2-8x+1)}{(1-x)(6x^6-38x^5+65x^4-60x^3+32x^2-9x+1)}
\]

and the number of hereditary semiorders on \( n \) elements is approximately

\( 3.3704^n \).
Hereditary Semiorders

\[
\begin{align*}
&\phantom{\text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1} \\
&\text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1, \text{C}_1 \end{align*}
\]
Corollary: (Remmel and Y. ‘12+) The collection of hereditary semiorders can be described by an list of meta-blocks* together with a collection of strong and weak boundaries*.
Hereditary Semiorders

**Corollary:** (Remmel and Y. ‘12+) The collection of hereditary semiorders can be described by an list of meta-blocks* together with a collection of strong and weak boundaries*.
In principle this should allow the height of hereditary semiorders to be determined.
Remaining Semiorders
Open Questions

height?

width?

dimension?

maximal elements?

Is $\Psi$ even the right bijection?