1 - Introduction to Combinatorics

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Distinguishing Qualities of Combinatorics

Problems in combinatorial mathematics tend to be easy to state and often involve concepts and structures that are relatively simple in nature. On the other hand, many of these problems have proven notoriously difficult to solve.

On the slides to follow, we give a quick sampling of such problems. No definitions are given, but students should be able to figure out what is being discussed. Size alone can make an elementary problem very difficult. But some problems of modest size are easy and some are hard. We want to be able to tell the difference ... and sometimes the distinction is very subtle.
Are Two Objects Identical?

Are these two sequences the same?

\[ S = 101011011010111011000011000101010100 \]

\[ T = 101011011010111010000011000101010100 \]
Are these Integers the Same?

124473218742449973221532177439883283\n28488764932089439237759922275943754\n2938745304583050720200061133612222222\n28475502020222022202225375757567363

1244732187424499732215321774398\n83283284887649320894392377\n59922275943754293874530458305072020\n006113361222222228476502020222202\n2202225375757567363
Question  I have two 1.44 mb floppy disks. Is it possible for me to determine - with absolute certainty - whether their contents are identical?

Remark  Some of you are too young to remember floppy disks, but if you will come to my office, I will show you one!

Question  I have two 4.4 gb dvd’s. Is it possible for me to determine - with absolute certainty - whether their contents are identical?
Adding Fractions

Remark In elementary school, students are taught to add fractions by finding least common multiples. For example,

\[
\frac{4}{15} + \frac{11}{21} = \frac{4}{3\cdot5} + \frac{11}{3\cdot7} = \frac{4\cdot7}{3\cdot5\cdot7} + \frac{11\cdot5}{3\cdot5\cdot7} = \frac{28 + 55}{105} = \frac{83}{105}
\]

Remark To add fractions, at least the way you were taught in elementary school, requires us to find the least common multiple of 15 and 21. This task is accomplished by factoring 15 = 3\cdot5 and 21 = 3\cdot7.

Question Is factoring easy or hard?
Adding Fractions

**Question**

How would you add the following fractions?

\[
\begin{align*}
&\frac{12167445732}{150072858731839} + \frac{10129488830173}{29133961727550713872} \\
\end{align*}
\]
Question  Is the integer $n$ shown below a prime?

\[ n = 127860603780100488460828699419348872552774882392154341226932424893189671902273018148382469 \]

Answer  Maple says “no”. In fact, in less than 3 seconds, Maple revealed that $n = p \times q$ where

\[ p = 364133217234400003717 \quad \text{and} \]
\[ q = 280829369862134719390036617067 \]

Maple also reported that both $p$ and $q$ are primes.
**Question**  
Is the integer $n$ shown below a prime?

$$n = 33319100065905904618088073943717337775739127140\backslash 72006500098512708917465853896616861419049384045100\backslash 6968337649739$$

**Answer**  
Maple did not give an answer, at least not after some 15 minutes of computing time. However, an oracle told me that $n$ is in fact the product of the following two primes.

$$p = 53542885039615245271174355315623704334284773568199$$

$$q = 622288097498926496141095869268883999563096063\backslash 592498055290461$$

Should I just have been more patient? Would Maple have eventually discovered this factorization of $n$?
Fair Division

Example  Given the numbers:

12  17  22  31  48

We observe that 12 + 22 + 31 = 17 + 48.

Question  Can you find a fair division of the numbers:

46  63  77  85  91  102  113  142  168  184  192  210
240  253  267  295  304  322  339  360  381  399
401  439  444  467  482  492  520  531  552
**Harder Problems**

**Question** I have a 1.44 mb floppy disk full of integers. Can I be certain that the total computing power on the planet is enough to settle the fair division problem for this set of numbers?

**Question** I have a 4.4 gb dvd full of integers. Can I be certain that the total computing power on the planet is enough to settle the fair division problem for this set of numbers?
Alice, Bob and Carlos

**Question** Carlos gives Alice and Bob a list of 10,000 integers, each of size at least 500 and at most 5,000,000. After examining the list and making several hours of computations, Alice says there is no fair division of the integers while Bob says the opposite. Carlos doesn’t know for sure who is right, but one of them will find it relatively easy to convince Carlos of the correctness of their answer. Which one?

**Question** Is the narrative the same if the list has 100,000,000,000 integers?
Graphs
Question How many vertices does this graph have? How many edges?

Remark In fact, there are 29 vertices, since the label 21 is mistakenly used twice!! So be careful!
Question  Can you draw this graph without crossings?
Question: What is the length of the shortest path from 1 to 12?
Question: Is this graph connected?
Question  Starting from 1, and walking only along edges, what is the maximum number of vertices you can visit without visiting any vertex more than once?
Slightly Harder Problem

1. What is a good way to convey essential information for a graph with 1,579,200 vertices?
2. Given such a graph, do you have any chance of determining whether it can be drawn without edge crossings?
3. Can you determine whether it is connected?
4. Can you find the maximum distance between two vertices?
5. Can you determine whether there is a way to visit each vertex exactly once, walking only on edges? Assume access to a super computer.
Question  Within five years of “getting out” from Georgia Tech, would you rather your annual salary in US dollars be $2^{10^4}$ or $100,000 	imes 100,000$?
Another Practical Problem

Question  Looking to the future, now that you have graduated and achieved considerable fame, success and wealth, your generous gift back to Georgia Tech (earmarked for the the School of Mathematics, of course) will be which of the following amounts in US dollars?

\[ 2^{102} \quad \text{or} \quad 1024 \times 1024 \]
Families of Disks
“Kissing Coins” Graphs
The Kissing Coins Theorem

**Theorem** (Koebe, 1936) Every graph that can be drawn without crossings has a representation as a set of "kissing coins".

Trust me, it’s true!!

**Question** Carlos has the data for a planar graph with 187,249 vertices. Is is reasonable that Xing can compute the values for a "kissing coins" representation of the graph?
1. Are there theorems that can be easily stated but for which any proof must be very long?
2. How can one easily distinguish between easy problems and hard problems?
3. In settling an argument, is it always the case that each side has an equal chance in convincing an impartial referee of the correctness of their opinion?
4. What is the precise meaning of the following words: Big, small, difficult, easy, doable, impossible, long, short.
**Question**  Dave found the data to define a graph with 500,000 vertices. He wondered whether this graph has a cycle which visits each vertex exactly once, returning at the end to the starting vertex. Xing loves computational challenges, and told Dave that he would have an answer after a weekend of computing. Yolanda was skeptical as she felt that the number of vertices was very, very large. Dave said that the data would easily fit on a 1.44 mb floppy disk, so it couldn’t be that hard. Zori didn’t see the relevance of the discussion and tuned out.