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1 - Introduction to Combinatorics

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Distinguishing Qualities of Combinatorics

Problems in combinatorial mathematics tend to be easy to state and often involve concepts and structures that are relatively simple in nature. On the other hand, many of these problems have proven notoriously difficult to solve.

On the slides to follow, we give a quick sampling of such problems. No definitions are given, but students should be able to figure out what is being discussed. Size alone can make an elementary problem very difficult. But some problems of modest size are easy and some are hard. We want to be able to tell the difference … and sometimes the distinction is very subtle.

Are Two Objects Identical?

Are these two sequences the same?

S = 1010110110101110110000011000101010100

T= 1010110110101110100000011000101010100

Are these Integers the Same?

```
12447321874244997322153217743988283\
28488764932089439237759922275943754\
29387453045830507202000611336122222\
2847550202022202222537575757567363

124473218742449973221532177439 8\
832832848887649320894392377\
5992227594375429387453045830507202 0\
0061133612222222284765020202222 02\
220222253757575 67363
```

Harder Problems

Question I have two 1.44 mb floppy disks. Is it possible for me to determine – with absolute certainty – whether their contents are identical?

Remark Some of you are too young to remember floppy disks, but if you will come to my office, I will show you one!

Question I have two 4.4 gb dvd's. Is it possible for me to determine – with absolute certainty – whether their contents are identical?

Adding Fractions

Remark In elementary school, students are taught to add fractions by finding least common multiples. For example,

$$\frac{4}{15}+\frac{11}{21}=\frac{4}{3\circ 5}+\frac{11}{3\circ 7}=\frac{4\circ 7}{3\circ 5\circ 7}+\frac{11\circ 5}{3\circ 5\circ 7}=\frac{28+55}{105}=\frac{83}{105}$$

Remark To add fractions, at least the way you were taught in elementary school, requires us to find the least common multiple of 15 and 21. This task is accomplished by factoring $15 = 3\circ 5$ and $21 = 3\circ 7$.

Question Is factoring easy or hard?

Adding Fractions

Question How would you add the following fractions?

$$\frac{12167445732}{15007285873183 9} + \frac{10129488830173}{291339617275507138 72}$$

Factoring

Question Is the integer n shown below a prime?

n = 12786060378010048846082869941934887255277748\
8239215434122693242489318967190227301814838 2469

Answer Maple says "no". In fact, in less than 3 seconds, Maple revealed that $n = p*q$ where

p = 3641332172344000 3717 and

q = 28082936986213471939003661 7067

Maple also reported that both p and q are primes.

Same problem – with bigger numbers

Question Is the integer n shown below a prime?

n = 333191000659059046180880739437173377757391271 40\
720065000985127089174658538966168614190493840451 00\
6968337649739

Answer Maple did not give an answer, at least not after some 15 minutes of computing time. However, an oracle told me that n is in fact the product of the following two primes.

p = 5354288503961524527117435531562370433428477356 8199

q = 6222880974989264961410958692688839995630960 63\
592498055290461

Should I just have been more patient? Would Maple have eventually discovered this factorization of n?

Fair Division

Example Given the numbers:

12 17 22 31 48

We observe that $12 + 22 + 31 = 17 + 48$.

Question Can you find a fair division of the numbers:

46 63 77 85 91 102 113 142 168 184 192 210
240 253 267 295 304 322 339 360 381 399
401 439 444 467 482 492 520 531 552

Harder Problems

Question I have a 1.44 mb floppy disk full of integers. Can I be certain that the total computing power on the planet is enough to settle the fair division problem for this set of numbers?

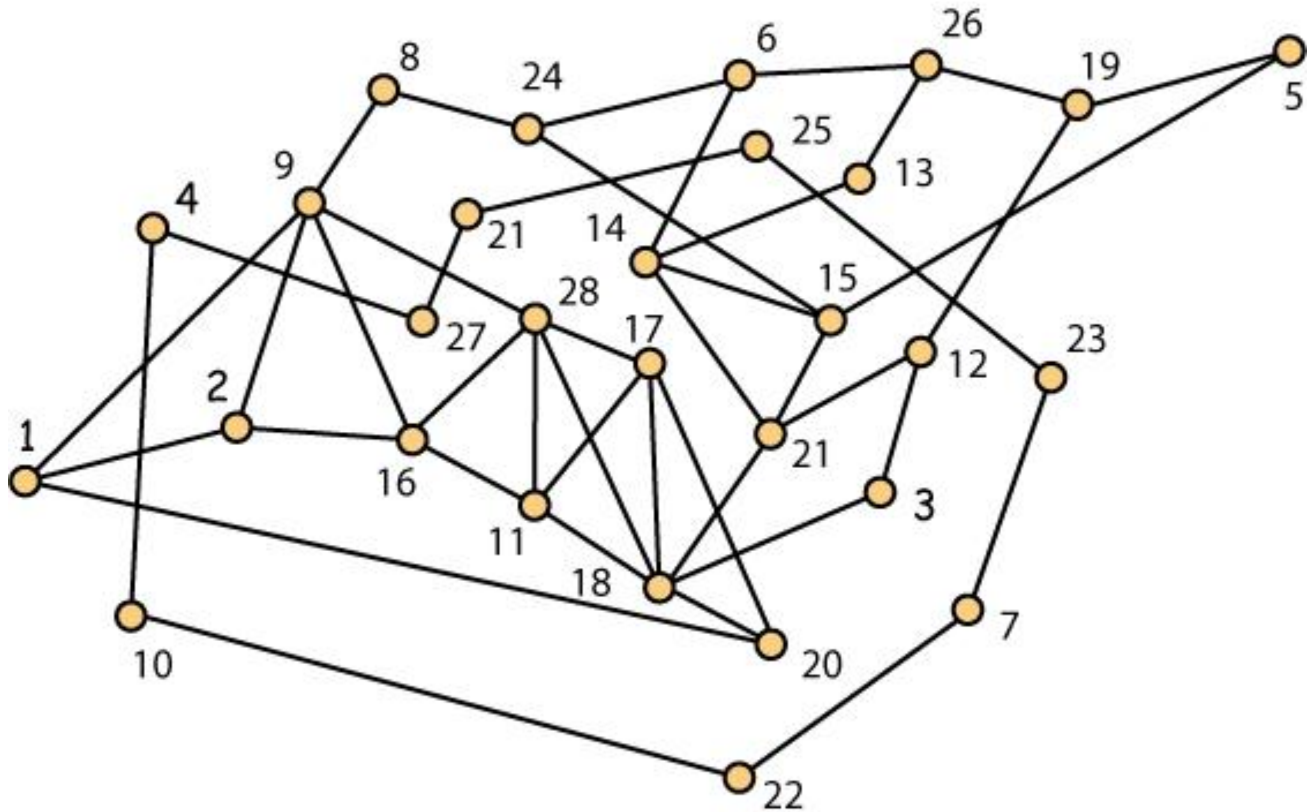
Question I have a 4.4 gb dvd full of integers. Can I be certain that the total computing power on the planet is enough to settle the fair division problem for this set of numbers?

Alice, Bob and Carlos

Question Carlos gives Alice and Bob a list of 10,000 integers, each of size at least 500 and at most 5,000,000. After examining the list and making several hours of computations, Alice says there is no fair division of the integers while Bob says the opposite. Carlos doesn't know for sure who is right, but one of them will find it relatively easy to convince Carlos of the correctness of their answer. Which one?

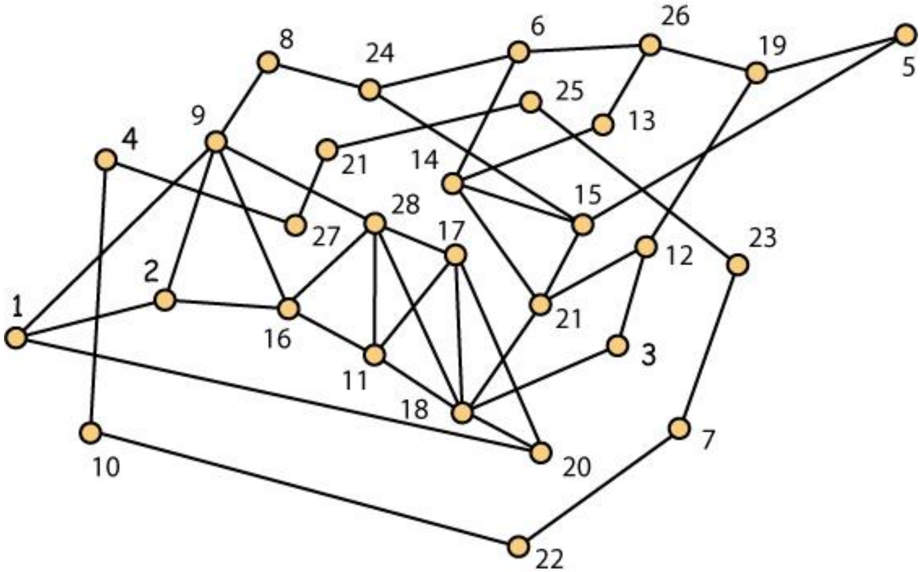
Question Is the narrative the same if the list has 100,000,000,000 integers?

Graphs



Natural Questions

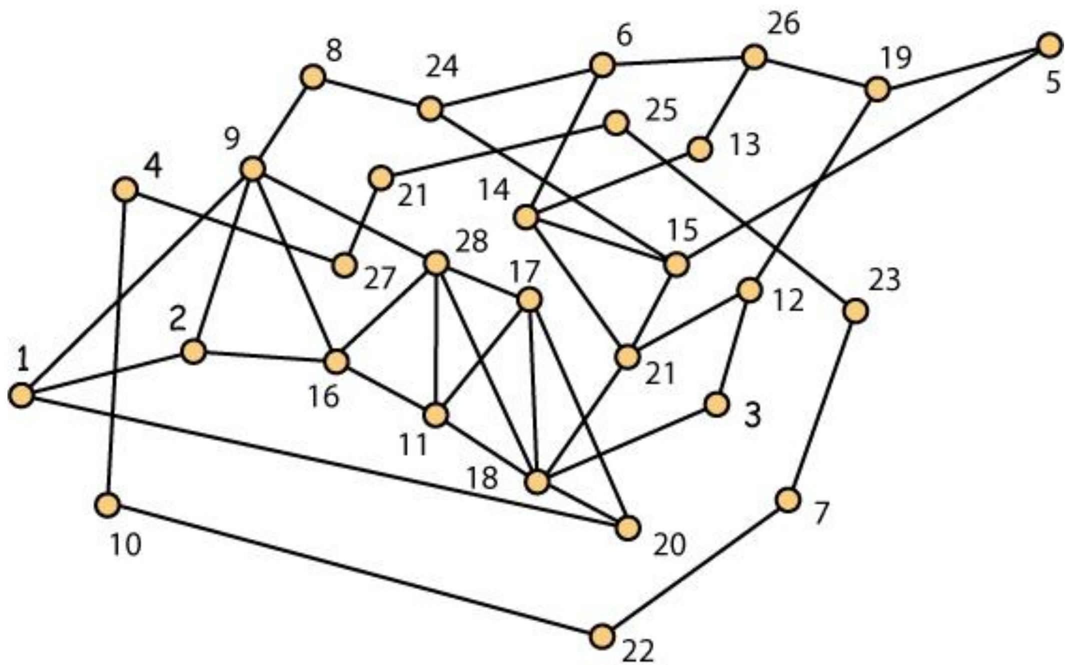
Question How many vertices does this graph have? How many edges?



Remark In fact, there are 29 vertices, since the label 21 is mistakenly used twice!! So be careful!

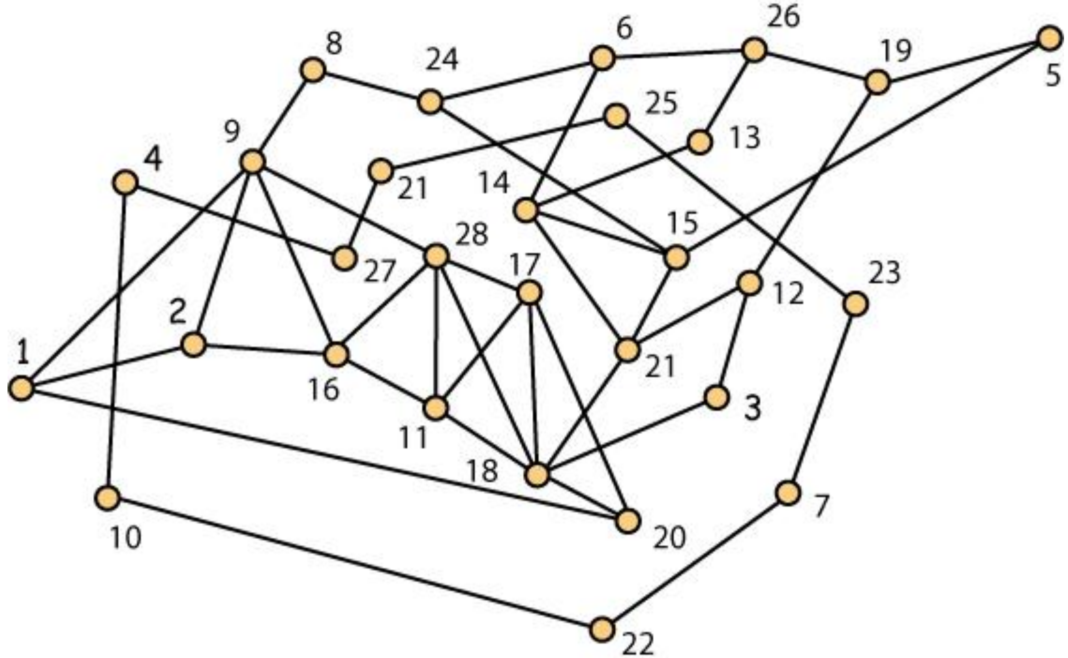
Natural Questions

Question Can you draw this graph without crossings?



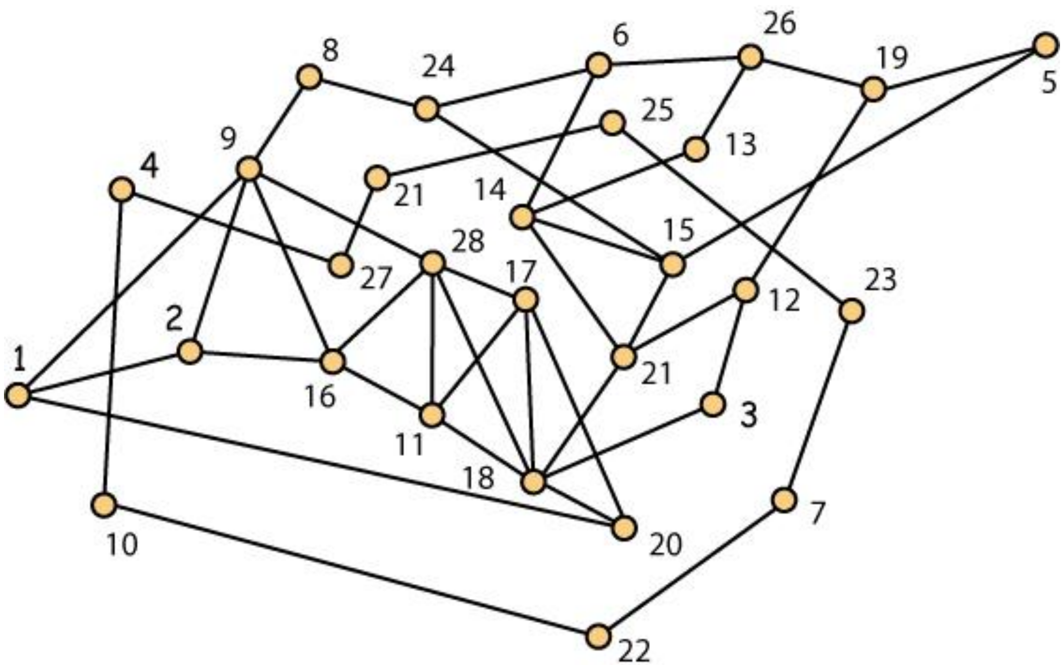
Natural Questions

Question What is the length of the shortest path from 1 to 12?



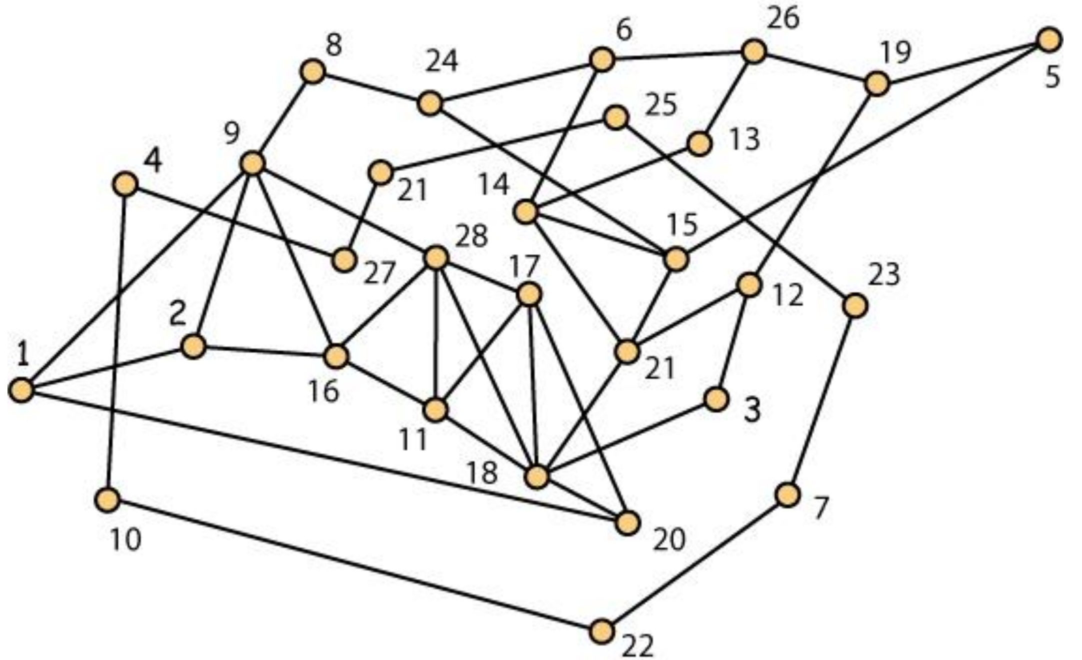
Natural Questions

Question Is this graph connected?



Natural Questions

Question Starting from 1, and walking only along edges, what is the maximum number of vertices you can visit without visiting any vertex more that once?



Slightly Harder Problem

1. What is a good way to convey essential information for a graph with 1,579,200 vertices?
2. Given such a graph, do you have any chance of determining whether it can be drawn without edge crossings?
3. Can you determine whether it is connected?
4. Can you find the maximum distance between two vertices?
5. Can you determine whether there is a way to visit each vertex exactly once, walking only on edges? Assume access to a super computer.

Very Practical Problem

Question Within five years of "getting out" from Georgia Tech, would you rather your annual salary in US dollars be

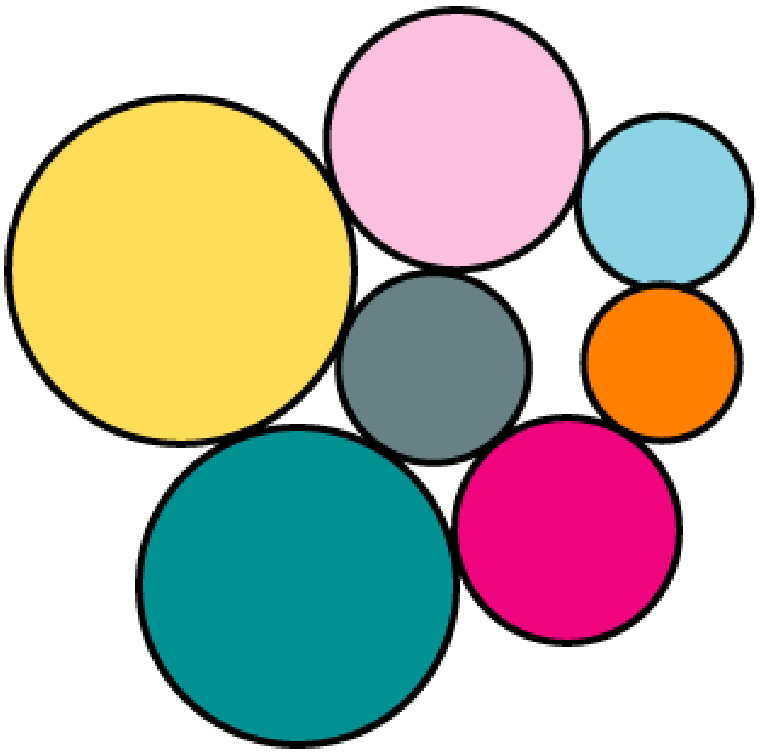
2^{10^4} or $100{,}000 \times 100{,}000$?

Another Practical Problem

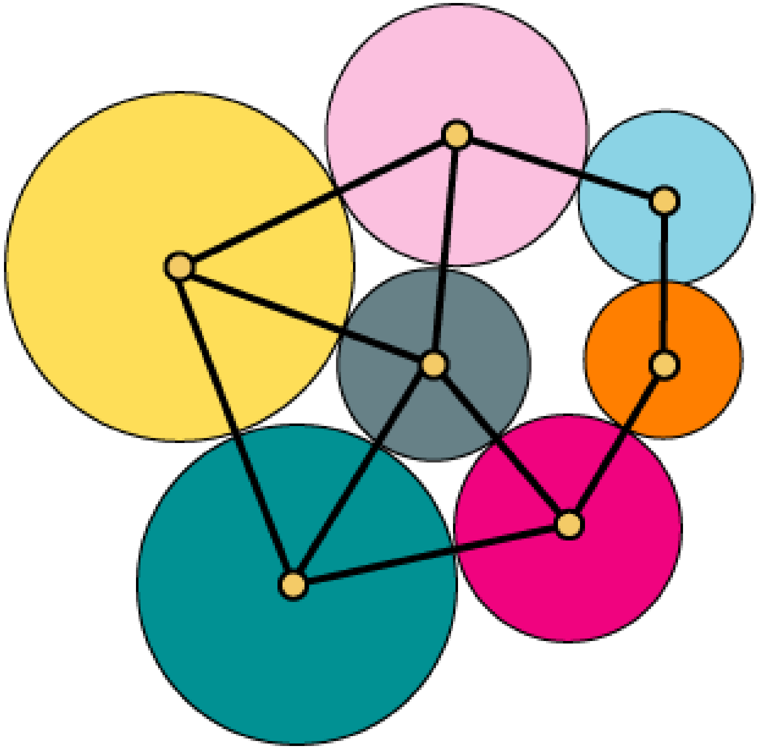
Question Looking to the future, now that you have graduated and achieved considerable fame, success and wealth, your generous gift back to Georgia Tech (earmarked for the the School of Mathematics, of course) will be which of the following amounts in US dollars?

2^{10^2} or 1024×1024

Families of Disks



"Kissing Coins" Graphs



The Kissing Coins Theorem

Theorem (Koebe, 1936) Every graph that can be drawn without crossings has a representation as a set of ``kissing coins".

Trust me, it's true!!

Question Carlos has the data for a planar graph with 187,249 vertices. Is is reasonable that Xing can compute the values for a "kissing coins" representation of the graph?

Tantalizing Questions

1. Are there theorems that can be easily stated but for which any proof must be very long?
2. How can one easily distinguish between easy problems and hard problems?
3. In settling an argument, is it always the case that each side has an equal chance in convincing an impartial referee of the correctness of their opinion?
4. What is the precise meaning of the following words: Big, small, difficult, easy, doable, impossible, long, short.

Dave, Xing, Yolanda and Zori

Question Dave found the data to define a graph with 500,000 vertices. He wondered whether this graph has a cycle which visits each vertex exactly once, returning at the end to the starting vertex. Xing loves computational challenges, and told Dave that he would have an answer after a weekend of computing. Yolanda was skeptical as she felt that the number of vertices was very, very large. Dave said that the data would easily fit on a 1.44 mb floppy disk, so it couldn't be that hard. Zori didn't see the relevance of the discussion and tuned out.