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12 - Cover Graphs and Comparability Graphs

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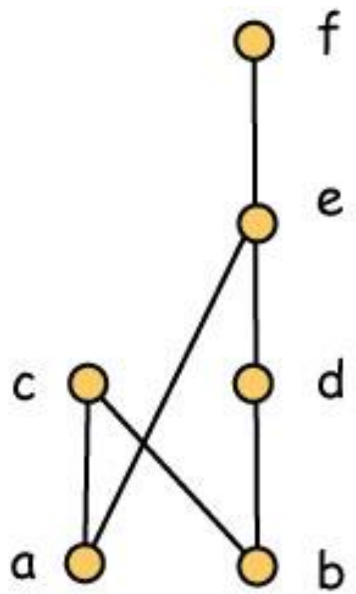
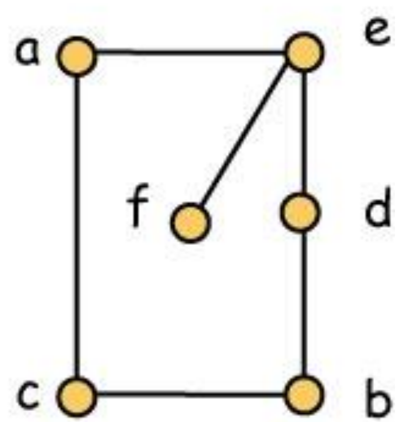
Cover Graphs

Definition A graph G is a cover graph when there is a poset P on the same ground set so that G is the cover graph of P.





Detecting Cover Graphs

Question Given a graph G, how hard is it to determine whether G is a cover graph?

Observation This problem is in NP, since there is an easily testable certificate for a "yes" answer.

Observation In certain very special circumstances, there is a certificate for a "no" answer. For example, the answer is "no" when the graph contains a triangle.

A Hard Problem (At least we think so)

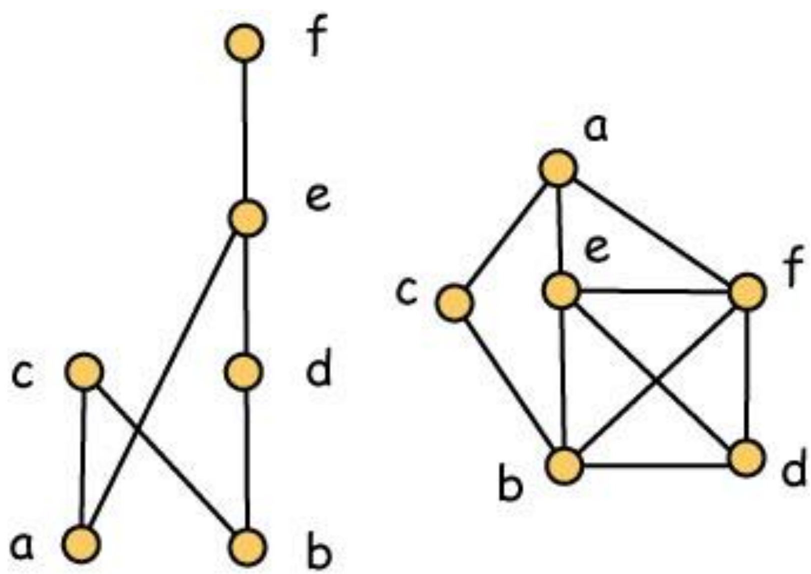
Definition A graph G is called a cover graph if there is a poset P whose cover graph is G.

Theorem (Brightwell 1993, Nešetřil and Rödl 1993) The question "Is G a cover graph?" is NP-complete.

Observation Unless we can crack the infamous $P = NP$? problem, we have no hope of answering efficiently whether a given graph is a cover graph.

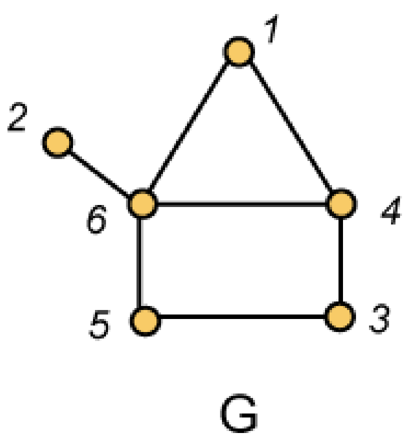
Comparability Graphs

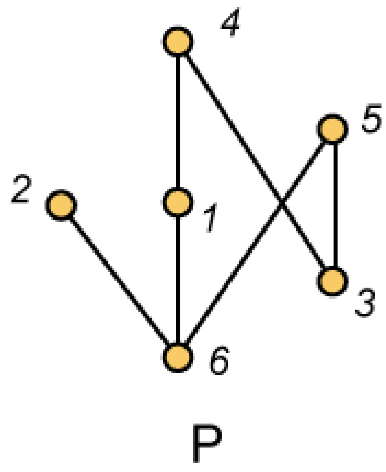
Definition A graph G is a comparability graph when there is a poset P on the same ground set so that G is the comparability graph of P.



Comparability Graphs (2)

Example A graph *G* shown below on the left is a comparability graph as evidenced by the poset on the right. Note that this poset has the same cover graph as the poset on the preceding slide, but their comparability graphs are different.





Detecting Comparability Graphs

Question Given a graph G, how hard is it to determine whether G is a comparability graph?

Observation This problem is in NP, since there is an easily testable certificate for a "answer."

Observation In certain very special circumstances, there is a certificate for a "no" answer. For example, the odd cycle C_5 is not a comparability graph.

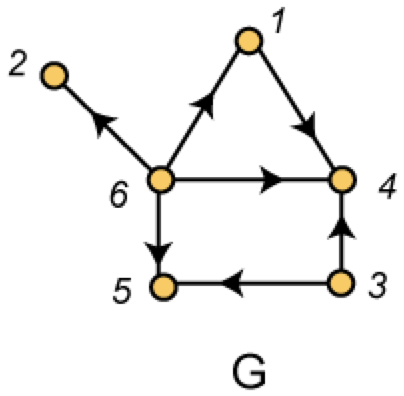
An Easy Problem (We know it is!)

Definition A graph G is called a comparability graph if there is a poset P whose cover graph is G.

Theorem (Discovered by many researchers) The question "Is G a comparability graph?" is in the class P, i.e., there is a polynomial time algorithm which will settle the issue completely.

Transitive Orientations

Alternate Definition A graph G is a comparability graph if G can be transitively oriented, i.e., if there is a directed edge from x to y and a directed edge from y to z, then xz is an edge in G and it is directed from x to z.



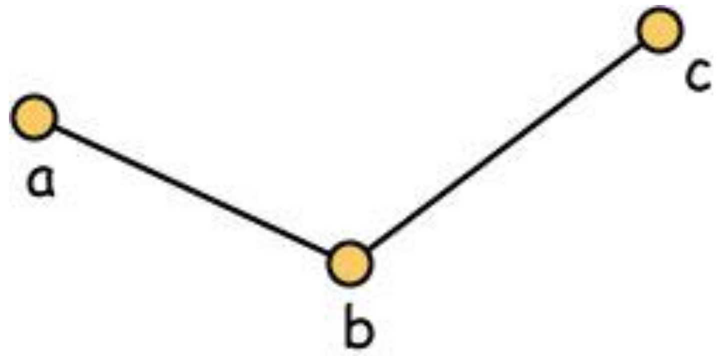
A Notational Convention

Remark In graph theory, when researchers speak of a directed graph, they normally allow for distinct vertices x and y an edge from x to y as well as an edge from y to x. On the other hand, when speaking of an oriented graph, given distinct vertices x and y, there is at most one edge with x and y as end points.

The P_3 Rule (also called the "Vee" Rule)

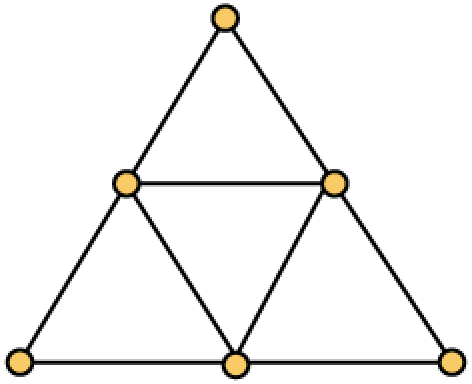
Observation In a transitive orientation of G, when $\{a, b, c\}$ induces a path P_3 as shown below, then either:

1. Both edges are oriented towards b, or
2. Both edges are oriented away from b.



Forbidden Graphs

Exercise Evidently, if G is a comparability graph, then so is every induced subgraph of G. However, the graph shown below is not a comparability graph. However, delete any vertex and the remaining graph is then a comparability graph.



The Algorithm

Initialization Choose an unoriented edge and assign it a direction.

Update Use the P_3 rule repeatedly to force additional directions on previously unoriented edges.

Loop If no additional forces and no conflicts, return to Initialization step.

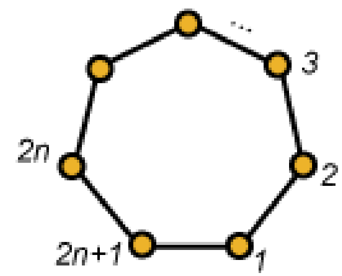
Halting Rule Quit with "no" answer if you get a conflict. Quit with yes answer if all edges are oriented and there are no conflicts.

Gallai's Theorem

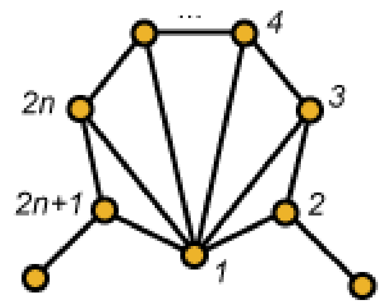
Theorem (T. Gallai, 1967) A graph G is a comparability graph if and only if it does not contain as an induced subgraph any of the graphs shown in Part I or the complements of any of the graphs shown in Part II.

Remark You are invited to contrast this result with, for example, Kuratowski's theorem, where by comparison, the number of forbidden structures is infinite but all are based on two simple examples, K_5 and $K_{3,3}$. Here it is quite an accomplishment that Gallai was able to complete the proof in a finite number of pages.

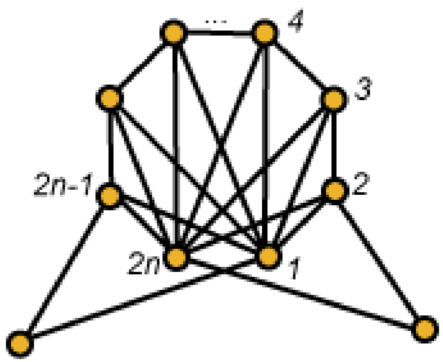
Families of Forbidden Graphs - I



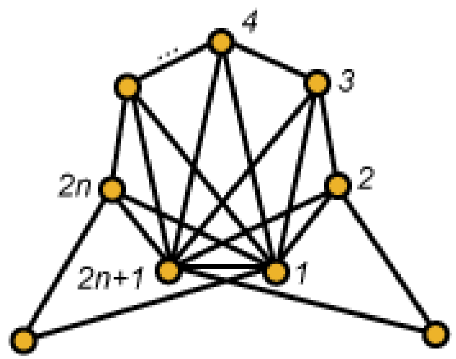
C_{2n+1}; $n \ge 2$



J_n; $n \ge 2$



J'_n; $n \ge 3$



J''_n; $n \ge 2$

Families of Forbidden Graphs - II